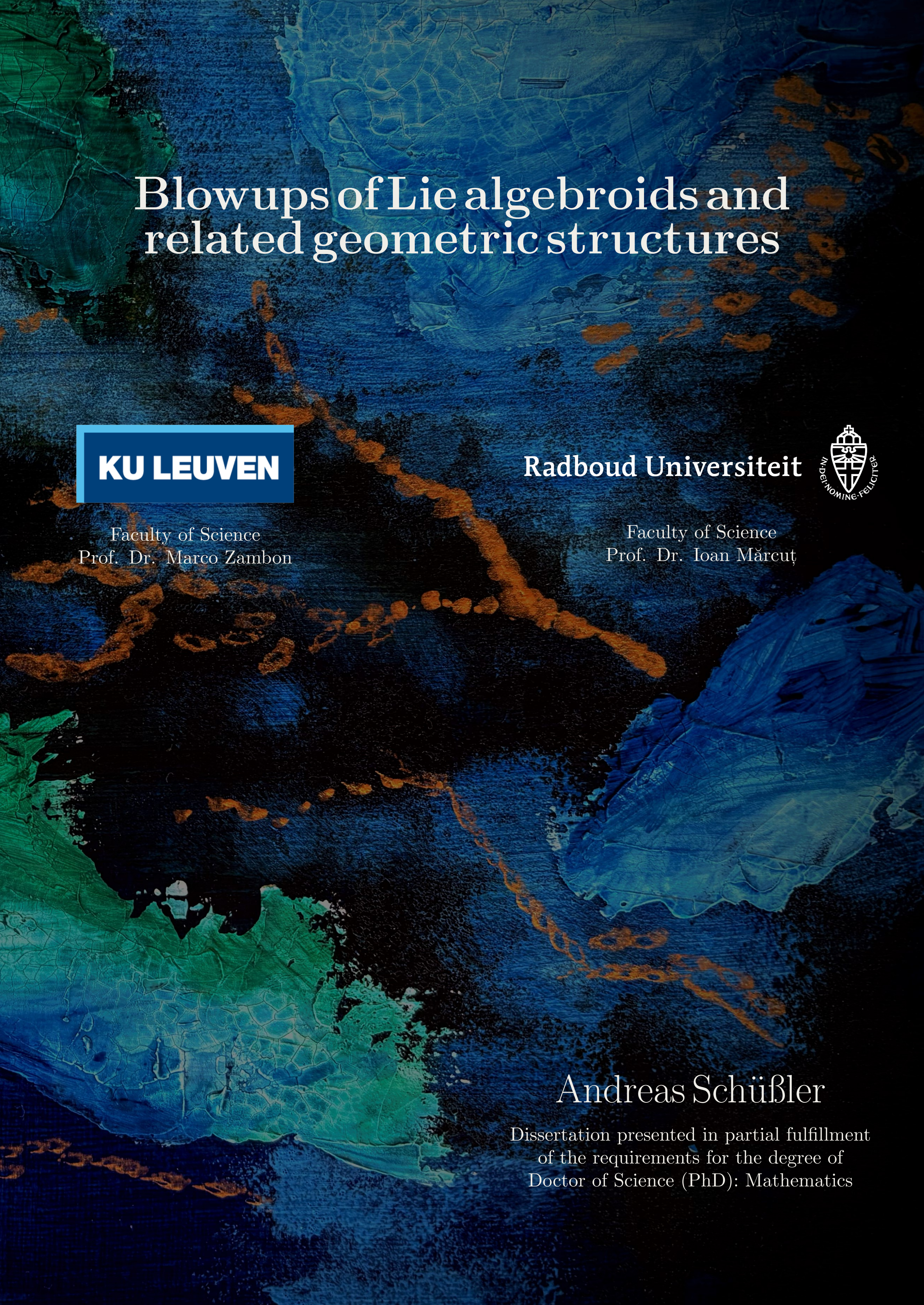

Blowups of Lie algebroids and related geometric structures

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Contents

Abstract

The general themes of this thesis are Lie algebroid cohomology and lifts of certain geometric structures appearing in Poisson geometry to real projective blowups. The thesis consists of five articles.

In the first article, we develop a Serre spectral sequence for Lie algebroids associated to the inclusion of a Lie subalgebroid. This construction reproduces and slightly generalises several known spectral sequences in the context of Lie algebroids, such as the Leray-Serre spectral sequence for de Rham cohomology associated to a fibration, or the Mackenzie spectral sequence for Lie algebroid extensions. Moreover, we show that one can use the Serre spectral sequence for Lie algebroids to compute formal Lie algebroid cohomology around submanifolds.

In the second article, we discuss real projective blowups of Lie algebroids in the context of Lie algebroid cohomology. More precisely, we study the blowdown map on the level of cohomology. We then apply our findings to generalise the Mazzeo-Melrose result on b-cohomology by computing the Lie algebroid cohomology of the blowup of an arbitrary transversal. Further, we reproduce the known result on the Lie algebroid cohomology of $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ using blowups. Moreover, we use similar techniques as developed for the blowdown map to compute the de Rham cohomology of real projective blowups.

In the third article, we address the following question. Given a submanifold of a Dirac manifold, under which circumstances does there exist a Dirac structure on the real projective blowup which extends the original Dirac structure? And, if such a structure exists, when is the blowdown map forward or backward Dirac? We give a complete answer to this question, which generalises a result by Polishchuk on lifting Poisson structures in this context. Moreover, for the main theorem we prove a classification result on Lie algebras, which might be of independent interest.

The fourth article contains a proof of the aforementioned result by Polishchuk within the differential-geometric framework. In particular, we derive the form of the linearisation of the Poisson structure as stated by Gualieri and Li, and give an interpretation of the explicit form of the linearisation.

The fifth article consists of two parts. In the first part, we examine the relation between the notions of pullbacks under transverse maps of Dirac structures, Lie algebroids, and their singular foliations. In the second part, we start with a Lie algebroid and a transverse submanifold. There are several ways to produce a Lie algebroid structure over the real projective blowup of the base: the pullback by the blowdown map and the blowup of a Lie subalgebroid supported over the submanifold. We see that those Lie algebroids are not isomorphic, and compute the singular foliation of some of them.

Dutch version

Dit proefschrift bestaat uit drie artikelen.

In het eerste artikel ontwikkelen we een Serre spectraalrij voor Lie algebroïden die geassocieerd worden met de inclusie van een Lie subalgebroïde. Deze constructie reproduceert en veralgemeniseert enkele bekende spectraalrijen op enkele fronten in de context van Lie algebroïden. Bijvoorbeeld de Leray-Serre spectraalrij voor de de Rham cohomologie die geassocieerd wordt met een filtratie, of de Mackenzie spectraalrij voor Lie algebroïde uitbreidingen. Bovendien laten we zien dat de Serre spectraalrijen voor Lie algebroïden gebruikt kunnen worden om formele Lie algebroïde cohomologieën rond subvariëteiten te berekenen.

In het tweede artikel bespreken we reële projectieve opblazing van Lie algebroïden in de context van Lie algebroïdische cohomologie. Om in detail te treden bestuderen wij de afblazende afbeelding op cohomologisch niveau. We passen onze bevindingen dan toe om de Mazzeo-Melrose resultaten over b-cohomologie te generaliseren door middel van het berekenen van de Lie algebroïdische cohomologie van de opblazing van een willekeurige transversaal. Ook reproduceren we het bekende resultaat over de Lie algebroïdische cohomologie van $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ met behulp van opblazingen. Daarnaast gebruiken we vergelijkbare technieken die ontwikkeld zijn voor de afblazende map om de de Rham cohomologie van een reële projectieve opblazing te kunnen berekenen.

In het derde artikel bespreken we de volgende vraag: gegeven een subvariëteit van een Dirac variëteit, onder welke voorwaarden bestaat er een Dirac structuur op de reële projectieve opblazing zodat ook de oorspronkelijke Dirac structuur uitbreidt? En als zo'n structuur bestaat, wanneer is de afblazende afbeelding voorwaarts of achterwaarts Dirac? We geven een volledig antwoord op deze vraag, wat tevens een veralgemenisering is van een resultaat van Polishchuk over het liften van Poisson structuren in deze context. Bovendien, als hoofdstelling bewijzen we een classificatie van Lie algebras, wat mogelijk op zichzelf al interessant zou kunnen zijn.

Het vierde artikel bevat een bewijs van het hiervoor genoemde resultaat van Polishchuk vanuit een differentiaal meetkundig oogpunt. In het bijzonder leiden we de vorm van de linearisatie af van de Poisson structuur zoals gegeven is door Gualieri en Li, en we geven een interpretatie van de expliciete vorm van de linearisatie.

Het vijfde artikel bestaat uit twee delen. In het eerste deel bestuderen we de relatie tussen de noties van een pullback onder transversale maps van Dirac structuren, Lie algebroïden, en hun foliaties. In het tweede deel starten we met een Lie algebroïde en een transversale variëteit. Er zijn enkele manieren om een Lie algebroïdische structuur over de reële projectieve opblazing van de basis te produceren: de pullback van de afgeblazen map, en de opblazing van een Lie subalgebroïde met support over de subvariëteit. We merken dat deze Lie algebroïdes niet isomorf zijn, en berekenen de singuliere foliatie van enkele.

Chapter 1

Introduction

In Section 1.1 we introduce some mathematical background to be able to formulate the main results of the following chapters, which we do in Section 1.2.

1.1 Preliminaries

We briefly introduce Poisson structures, Lie algebroids, and Dirac structures. These structures are closely related (Poisson structures induce both a Lie algebroid and a Dirac structure, every Dirac structure can be seen as a Lie algebroid). Moreover, we outline real projective blowups for smooth manifolds, a construction which is fundamental for most parts of this thesis.

1.1.1 Poisson structures

Let M be a manifold. A **Poisson structure**, introduced by Lichernowicz in [Lic77], on M is a bivector field $\pi \in \Gamma(\wedge^2 TM)$ that satisfies

$$[\![\pi, \pi]\!]_{\mathrm{S}} = 0,$$

where $[\![\,\cdot\,,\,\cdot\,]\!]_{\mathrm{S}}$ denotes the Schouten-Nijenhuis bracket on $\Gamma(\wedge^\bullet TM)$. Equivalently, the map

$$\begin{aligned} \{\,\cdot\,,\,\cdot\,\}\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) &\to \mathscr{C}^\infty(M) \\ (f, g) &\mapsto \pi(\mathrm{d}f, \mathrm{d}g) \end{aligned} \tag{1.1}$$

defines a Poisson bracket on $\mathscr{C}^\infty(M)$, i.e. it is $\mathbb{R}$-bilinear, skew-symmetric and, for all $f \in \mathscr{C}^\infty(M)$, the map

$$\{f,\,\cdot\,\}\colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M) \tag{1.2}$$

is a derivation for both the pointwise product as well as the bracket $\{\,\cdot\,,\,\cdot\,\}$ itself.

As any bivector field, π induces a map

$$\begin{aligned} \pi^\sharp \colon T^*M &\to TM \\ \alpha &\mapsto \mathrm{i}_\alpha \pi, \end{aligned} \tag{1.3}$$

where i_α denotes the left insertion of α into a multivector field.

Example 1.1.1 1. Let (M, ω) be a **symplectic manifold**, i.e. $\omega \in \Omega^2(M)$ is closed and nondegenerate. Then

$$\{f, g\} := X_g(f), \tag{1.4}$$

where $\mathrm{d}g = \omega(X_g, \,\cdot\,)$, defines a Poisson structure on M. When writing $\omega^\flat \colon TM \to T^*M$ for the induced map (defined analog to (1.3)), we have $(\omega^\flat)^{-1} = -\pi^\sharp$.

2. Let $\mathfrak{g}$ be a Lie algebra. Then there exists a canonical linear Poisson structure on $\mathfrak{g}^*$ uniquely defined by the following. For two linear functions $\xi, \eta \in \mathfrak{g} = (\mathfrak{g}^*)^*$ on $\mathfrak{g}^*$, the Poisson bracket is given by

$$\{\xi, \eta\}(\alpha) := \alpha([\xi, \eta]). \tag{1.5}$$

A Poisson manifold (M, π) carries a **singular symplectic foliation** given by $\pi^\sharp(\Gamma_\mathrm{c}(T^*M))$, which induces a decomposition into immersed, connected submanifolds endowed with symplectic structures,

$$M = \bigsqcup_S (S, \omega_S),$$

whose members are called **symplectic leaves**. The singular symplectic foliation is determined by the following conditions (see e.g. [CFM21, Proposition 1.8 and Theorem 4.1])

$$T_xS = \mathrm{Im}(\pi^\sharp_x \colon T^*_xM \to T_xM), \quad \omega_S(\pi^\sharp\alpha, \pi^\sharp\beta) = -\pi(\alpha, \beta), \quad \text{for all } x \in S.$$

Example 1.1.2 For a Lie algebra $\mathfrak{g}$, the symplectic foliation on $\mathfrak{g}^*$ is given by the orbits of the coadjoint action of $\mathfrak{g}$, see e.g. [DZ05, Theorem 1.5.8].

A **Poisson submanifold** $N \subseteq M$ is a submanifold such that $\pi\big|_N \in \Gamma(\wedge^2 TN)$. In this case, at every point $q \in N$ the conormal space $(T_qN)^{\mathrm{ann}}$ carries a Lie algebra structure, turning $(TN)^{\mathrm{ann}}$ into a bundle of Lie algebras. The Lie bracket on $(T_qN)^{\mathrm{ann}}$ is defined by

$$[(\mathrm{d}f)\big|_N(q), (\mathrm{d}g)\big|_N(q)] := (\mathrm{d}\{f, g\})\big|_N(q), \tag{1.6}$$

where f, g are smooth functions on M vanishing on N. The corresponding linear Poisson structure π_{lin} on the dual $\nu_N(M) := TM\big|_N/TN$ is, on $\nu_N(M)\big|_q$, the linearisation at q of the transverse component of the Poisson structure.

1.1.2 Lie algebroids

Lie algebroids [CF11, HM90, Mac05] are a generalisation of Poisson structures, as well as Lie algebras and tangent bundles, see Example 1.1.3. A **Lie algebroid** A over a base manifold M, denoted by $A \Rightarrow M$, consists of

- a vector bundle over a manifold, $A \to M$,
- a vector bundle map $\sharp\colon A \to TM$ covering the identity, called the **anchor map**,
- a Lie algebra structure $[\cdot\,,\cdot]$ on $\Gamma(A)$, the set of smooth sections of $A \to M$,

which satisfies the following compatibility condition, called the Leibniz identity,

$$[\alpha, f\beta] = f[\alpha,\beta] + (\mathscr{L}_{\sharp\alpha} f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$ and $f \in \mathscr{C}^\infty(M)$. Here, $\mathscr{L}_X f$ denotes the Lie derivative of a function f along a vector field X. For more on the general theory, see e.g. [Mac05, CF11, Mei17, CFM21].

A **representation** of a Lie algebroid $A \Rightarrow M$ on a vector bundle $V \to M$ is defined as a flat A-connection on V, i.e. a bilinear operator,

$$\nabla\colon \Gamma(A) \times \Gamma(V) \to \Gamma(V), \quad (\alpha, \eta) \mapsto \nabla_\alpha \eta,$$

which is $\mathscr{C}^\infty(M)$-linear in the first entry and satisfies the Leibniz identity in the second entry

$$\nabla_{f\alpha}\eta = f\nabla_\alpha\eta, \qquad \nabla_\alpha(f\eta) = f\nabla_\alpha\eta + (\mathscr{L}_{\sharp\alpha} f)\eta, \tag{1.7}$$

and has trivial curvature

$$\nabla_\alpha\nabla_\beta\eta - \nabla_\beta\nabla_\alpha\eta = \nabla_{[\alpha,\beta]}\eta, \tag{1.8}$$

where $f \in \mathscr{C}^\infty(M)$, $\alpha, \beta \in \Gamma(A)$, and $\eta \in \Gamma(V)$.

The notion of Lie algebroids unifies several concepts in differential geometry.

Example 1.1.3

1. A Lie algebra $\mathfrak{g}$ is the same as a Lie algebroid over a point, $A \Rightarrow \{*\}$, and the corresponding notions of representations coincide.
2. For a manifold M the tangent bundle TM is a Lie algebroid with bracket the usual Lie bracket of vector fields and anchor $\sharp = \mathrm{id}_{TM}$. A representation of $TM \Rightarrow M$ is the same as a vector bundle $V \to M$ endowed with a flat connection.
3. By the Frobenius Theorem, a regular foliation on a manifold M is the same as a Lie subalgebroid (in the sense below) of $TM \Rightarrow M$.

4. A Poisson structure $\{\cdot, \cdot\}$ on $\mathscr{C}^\infty(M)$ gives rise to a Lie algebroid structure on $T^*M \Rightarrow M$ with anchor $\sharp = \pi^\sharp$ and bracket on exact one-forms given by
$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\}, \tag{1.9}$$
where $f, g \in \mathscr{C}^\infty(M)$.

5. An infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M gives rise to a so-called action Lie algebroid, $\mathfrak{g} \ltimes M \Rightarrow M$. As a vector bundle, $\mathfrak{g} \ltimes M$ is the trivial bundle $\mathfrak{g} \times M \to M$. The bracket on constant sections is given by the bracket on $\mathfrak{g}$ and the anchor is the action map.

We recall the construction of **Lie algebroid cohomology**. To any Lie algebroid $A \Rightarrow M$, one associates the differential graded commutative algebra of de Rham forms on A,
$$\Big(\Omega^\bullet(A) := \bigoplus_{k=0}^{r} \Omega^k(A), \wedge, \mathrm{d}_A\Big), \quad \Omega^k(A) := \Gamma(\wedge^k A^*), \tag{1.10}$$
where $r = \mathrm{rank}(A)$ and d_A is defined similar to the usual de Rham exterior derivative
$$\begin{aligned}(\mathrm{d}_A\omega)(\alpha_0, \dots, \alpha_k) := &\sum_i (-1)^i \mathscr{L}_{\sharp\alpha_i} \omega(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k),\end{aligned} \tag{1.11}$$
where we regard elements in $\Omega^\bullet(A)$ as $\mathscr{C}^\infty(M)$-multilinear forms on sections $\alpha_i \in \Gamma(A)$ with values in $\mathscr{C}^\infty(M)$. Here, $\hat{\cdot}$ denotes the omission of the argument. By passing to cohomology one obtains the Lie algebroid cohomology of A, denoted by $\mathrm{H}^\bullet(A)$. More generally, given a representation $V \to M$ of A we have an induced differential on V-valued forms on A, which we continue denoting by d_A
$$\big(\Omega^\bullet(A, V) := \Gamma(\wedge^\bullet A^* \otimes V), \mathrm{d}_A\big), \tag{1.12}$$
defined by formula (1.11) with the operator ∇_{α_i} instead of $\mathscr{L}_{\sharp\alpha_i}$ (the latter corresponding to the canonical representation on $\mathbb{R} \times M$). Passing to cohomology, one obtains the Lie algebroid cohomology of $A \Rightarrow M$ with coefficients in the representation $V \to M$, denoted by $\mathrm{H}^\bullet(A, V)$.

Remark 1.1.4 Lie algebroid cohomology recovers many classical cohomology theories, e.g.

1. Chevalley-Eilenberg cohomology of a Lie algebra via $\mathfrak{g} \Rightarrow \{*\}$, as well as Lie algebra cohomology with coefficients in $\mathscr{C}^\infty(M)$ via $\mathfrak{g} \ltimes M \Rightarrow M$;

2. de Rham cohomology of a manifold via $TM \Rightarrow M$, as well as foliated cohomology;

3. Poisson cohomology of (M, π) via $T^*M \Rightarrow M$.

We also require the notion of a **generalised representation** [MS24]. A generalised representation of a Lie algebroid $A \Rightarrow M$ is a $\mathscr{C}^\infty(M)$-module $\mathfrak{M}$, endowed with a bilinear map

$$\nabla\colon \Gamma(A) \times \mathfrak{M} \to \mathfrak{M}$$

satisfying the axioms (1.7) and (1.8). Usual representations correspond to the case when $\mathfrak{M}$ is a finitely generated projective $\mathscr{C}^\infty(M)$-module.

To introduce $\mathfrak{M}$-valued cohomology, define the set of p-forms on A with values in $\mathfrak{M}$ as

$$\Omega^p(A,\mathfrak{M}) := \big\{\omega\colon \Gamma(A)^{\times p} \to \mathfrak{M} : \text{alternating and } \mathscr{C}^\infty(M)\text{-multilinear}\big\}. \tag{1.13}$$

The formula (1.11) generalises to this setting, and yields a differential

$$\mathrm{d}_A\colon \Omega^\bullet(A,\mathfrak{M}) \to \Omega^{\bullet+1}(A,\mathfrak{M}).$$

The associated cohomology groups will be denoted by

$$\mathrm{H}^\bullet(A,\mathfrak{M}).$$

For a classical representation V, the two equivalent notations $\mathrm{H}^\bullet(A,\Gamma(V)) = \mathrm{H}^\bullet(A,V)$ will, hopefully, not lead to confusions.

General **Lie algebroid morphisms** are introduced efficiently using the cochain complex of Lie algebroids. Let $A \Rightarrow M$ and $L \Rightarrow N$ be two Lie algebroids. A vector bundle map

$$\begin{tikzcd} L \arrow[r, "\Phi"] \arrow[d] & A \arrow[d] \\ N \arrow[r, "\phi"] & M \end{tikzcd}$$

induces a pullback homomorphism between the graded commutative algebras of de Rham forms $\Phi^*\colon (\Omega^\bullet(A),\wedge) \to (\Omega^\bullet(L),\wedge)$,

$$(\Phi^*\omega)_p(\alpha_1,\dots,\alpha_k) := \omega_{\phi(p)}(\Phi(\alpha_1),\dots,\Phi(\alpha_k)),$$

for all $p \in N$ and $\alpha_i \in L_p$. By definition, Φ is a Lie algebroid morphism if and only if Φ^* is a cochain map, i.e. for all $\omega \in \Omega^\bullet(A)$, we have

$$\Phi^*(\mathrm{d}_A\omega) = \mathrm{d}_L(\Phi^*\omega).$$

For example, for Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, this notion recovers Lie algebra maps $\Phi\colon \mathfrak{g} \to \mathfrak{h}$; Lie algebroid morphisms $\Phi\colon TM \to TN$ are determined by their base map, via $\Phi = T\phi$; Lie algebroid morphisms $\Phi\colon TM \to \mathfrak{g}$ are the same as flat, principal connections on the trivial principal G-bundle $G \times M$, where G is a connected Lie group integrating $\mathfrak{g}$.

Representations can be pulled back along a Lie algebroid morphism $\Phi\colon L \to A$. If $V \to M$ is a representation of $A \Rightarrow M$, the pullback representation of $L \Rightarrow N$ is on the **pullback vector bundle** $\phi^\sharp V \to N$ and is uniquely determined by the condition that the pullback map is a cochain map

$$\Phi^*\colon \big(\Omega^\bullet(A, V), \mathrm{d}_A\big) \to \big(\Omega^\bullet(L, \phi^\sharp V), \mathrm{d}_L\big).$$

A **Lie subalgebroid** of $A \Rightarrow M$ is a vector subbundle $L \hookrightarrow A$ over an injective immersion $N \hookrightarrow M$ endowed with a Lie algebroid structure $L \Rightarrow N$ (necessarily unique) for which the inclusion $i\colon L \hookrightarrow A$ is a Lie algebroid morphism.

The **pullback** of a Lie algebroid $A \Rightarrow M$ along a smooth map $f\colon N \to M$, called inverse image Lie algebroid in [HM90, Section 1], is given by

$$f^! A := \{(v, a) \in T_x N \oplus A_{f(x)} : T_x f(v) = \sharp a \in T_{f(x)} M,\ x \in N\}, \tag{1.14}$$

which carries a canonical Lie algebroid structure over N, provided it has constant rank. If this is the case, its anchor is the first projection $\sharp = \mathrm{pr}_1\colon f^! A \to TN$ and the second projection is a Lie algebroid map $f^! := \mathrm{pr}_2\colon f^! A \to A$ covering f. If $\iota\colon N \hookrightarrow M$ is a submanifold of M such that the inclusion is transverse to the anchor, N is called a **transversal** and we can consider $\iota^! A$ as a Lie subalgebroid of A. We use transversals to obtain Lie subalgebroids of A in Section 2.2.3 and Section 3.5, and the pullback via covering maps to compute Lie algebroid cohomology in Section 3.6.

If $N \subseteq M$ is a submanifold such that $\sharp(A\big|_N) \subseteq TN$, we call N an **invariant submanifold**, and the restriction $A\big|_N$ (which is the same as the pullback by the inclusion in the sense above) inherits a Lie algebroid structure with N as base, such that $A\big|_N \hookrightarrow A$ is a Lie algebroid morphism.

1.1.3 Dirac structures

Dirac structures were first introduced in [Cou90] to interpolate between Poisson structures and closed 2-forms on a manifold M. For more on the general theory, see e.g. [Bur13, CFM21].

Let M be a manifold. The **generalised tangent bundle**

$$\mathbb{T}M := TM \oplus T^*M \tag{1.15}$$

carries a nondegenerate, symmetric pairing

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle\colon \mathbb{T}M \times_M \mathbb{T}M &\to \mathbb{R} \\ \begin{pmatrix} v \\ \xi \end{pmatrix}, \begin{pmatrix} w \\ \eta \end{pmatrix} &\mapsto \frac{1}{2}(\xi(w) + \eta(v)) \end{aligned} \tag{1.16}$$

and a bracket, for any closed 3-form $H \in \Omega^3(M)$, defined by

$$\begin{aligned} [\,\cdot\,,\,\cdot\,]\colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) &\to \Gamma(\mathbb{T}M) \\ \begin{pmatrix} X \\ \alpha \end{pmatrix}, \begin{pmatrix} Y \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} [X, Y] \\ \mathscr{L}_X \beta - \mathrm{i}_Y \mathrm{d}\alpha + \mathrm{i}_Y \mathrm{i}_X H \end{pmatrix}, \end{aligned} \tag{1.17}$$

called the **Dorfman bracket**.

Remark 1.1.5 Although the bracket $[\cdot\,,\,\cdot]$ satisfies a Jacobi identity, it does not define a Lie bracket as it is not skew symmetric. One has

$$[A,B]-[B,A]=2\mathrm{d}\langle A,B\rangle \tag{1.18}$$

for $A,B\in\Gamma(\mathbb{T}M)$, where d denotes the de Rham differential.

Together with the **anchor map** $\sharp=\mathrm{pr}_{TM}\colon\mathbb{T}M\to TM$ these structure maps endow $\mathbb{T}M$ with the structure of a **twisted Courant algebroid**.

For a fixed closed 3-form $H\in\Omega^3(M)$, a subbundle $D\subset\mathbb{T}M$ is called an **H-twisted Dirac structure** on M if D is maximally isotropic (i.e. Lagrangian) with respect to the pairing $\langle\cdot\,,\,\cdot\rangle$, and involutive, i.e. $[\Gamma(D),\Gamma(D)]\subseteq\Gamma(D)$. We call D a **Dirac structure** if it is a twisted Dirac structure for $H=0$.

Standard examples for Dirac structures on M are given by the following (see e.g. [Gua11, Example 2.11-2.13]).

1. The graph
$$\mathrm{graph}(\pi)=\{(\pi^\sharp\xi,\xi)\in\mathbb{T}M:\xi\in T^*M\} \tag{1.19}$$
of a bivector field $\pi\in\Gamma(\wedge^2TM)$ is a Lagrangian subbundle of $\mathbb{T}M$, which is a Dirac structure iff π is Poisson, i.e. $[\![\pi,\pi]\!]_{\mathrm{S}}=0$.
2. The graph
$$\mathrm{graph}(\omega)=\{(v,\omega^\flat v)\in\mathbb{T}M:v\in TM\} \tag{1.20}$$
of a 2-form ω on M is Lagrangian, and defines a Dirac structure iff $\mathrm{d}\omega=0$ is closed.
3. More generally, graph(ω) for any $\omega\in\Omega^2(M)$ is a $\mathrm{d}\omega$-twisted Dirac structure.
4. The subbundle $T\mathcal{F}\oplus(T\mathcal{F})^{\mathrm{ann}}$ for a subbundle $T\mathcal{F}\subseteq TM$ is a Dirac structure iff $\mathcal{F}$ is a (regular) foliation.

Remark 1.1.6 By Remark 1.1.5, the bracket induced by the twisted Dorfman bracket on any Dirac structure is skew-symmetric. Hence, Dirac structures carry the structure of a Lie algebroid.

As morphisms between Dirac structures we consider forward and backward Dirac maps.

Definition 1.1.7 ([BR03, Wei82]) Let $\phi\colon X\to M$ be a smooth map. Let D_X be an H_X-twisted Dirac structure on X, and D_M be an H_M-twisted Dirac structure on M. If $H_X=\phi^*H_M$, we call ϕ

1. **a forward Dirac map**, if we have $(D_M)_{\phi(x)}=\mathfrak{F}_\phi((D_X)_x)$ for all $x\in X$, where
$$\mathfrak{F}_\phi((D_X)_x):=\left\{\begin{pmatrix}(T_x\phi)w\\ \xi\end{pmatrix}\in\mathbb{T}_{\phi(x)}M:\begin{pmatrix}w\\ (T_x\phi)^*\xi\end{pmatrix}\in(D_X)_x\right\};$$

2. **a backward Dirac map**, if we have $(D_X)_x = \mathfrak{B}_\phi((D_M)_{\phi(x)})$ for all $x \in X$, where

$$\mathfrak{B}_\phi((D_M)_{\phi(x)}) = \left\{ \begin{pmatrix} w \\ (T_x\phi)^*\xi \end{pmatrix} \in \mathbb{T}_x X : \begin{pmatrix} (T_x\phi)w \\ \xi \end{pmatrix} \in (D_M)_{\phi(x)} \right\}.$$

Forward Dirac maps generalise Poisson maps, while backward Dirac maps generalise pullbacks of 2-forms.

If $\iota\colon N \hookrightarrow M$ is a submanifold such that $\mathrm{pr}_{TM}(D\big|_N) \subseteq TN$ (i.e. N is an invariant submanifold for the induced Lie algebroid structure), we again call N invariant, and, just like in the Poisson case (1.6), there is an induced bundle of Lie algebras structure on the conormal bundle $(TN)^{\mathrm{ann}} \to N$. The Lie algebra structures come from the short exact sequence of Lie algebroids

$$0 \longrightarrow (TN)^{\mathrm{ann}} \longrightarrow D\big|_N \longrightarrow \mathfrak{B}_{\iota_N}(D) \longrightarrow 0. \tag{1.21}$$

1.1.4 Real projective blowups

Blowup constructions are well-known in algebraic geometry (see e.g. [Har77]). The idea is to replace a point or subvariety by all lines normal to it. A famous result by Hironaka [Hir64a, Hir64b] states that in characteristic zero, one can always desingularise algebraic varieties, i.e. obtain a smooth variety, by a sequence of blowups. In the context of smooth manifolds, a smooth submanifold is replaced by the projectivisation of its normal bundle, and singularities arise from additional geometric structures (a Poisson structure, a foliation, etc.).

As a set, the blowup $\mathrm{Blup}(M, N)$ of a closed and embedded submanifold N in M is given by

$$\mathrm{Blup}(M, N) = (M \setminus N) \cup \mathbb{P}(\nu_N(M)), \tag{1.22}$$

i.e. by replacing N with $\mathbb{P}(\nu_N(M))$, the projectivisation of the normal bundle

$$\nu_N(M) = TM|_N/TN$$

of N in M. There is a natural surjective map $p\colon \mathrm{Blup}(M, N) \to M$, called **blowdown map**, given by the fibre bundle projection on $\mathbb{P}(\nu_N(M))$ and by the identity on $M \setminus N$.

To give some intuition for the smooth structure on $\mathrm{Blup}(M, N)$, note first that the construction is local around N. Hence, it is enough to consider the blowup of the zero section N inside a vector bundle $E \to N$. Then

$$\mathrm{Blup}(E, N) \simeq \mathbb{L}(E), \tag{1.23}$$

where $\mathbb{L}(E)$ is the tautological line bundle over the projectivisation of E, i.e. it is the line bundle over $\mathbb{P}(E)$ with fibres given by

$$\mathbb{L}(E)_{[v]} = \{v' \in E : v' = \lambda v \text{ for some } \lambda \in \mathbb{R}\} \tag{1.24}$$

for $v \in E \setminus N$. In this setting, the blowdown map is given by $v'_{[v]} \mapsto v'$.

Remark 1.1.8 Given a submanifold chart $(U,(x,y))$ for N, i.e. $U\cap N=\{x=0\}$, one can construct charts for $p^{-1}(U)\subseteq \mathrm{Blup}(M,N)$. For $i=1,\dots,\mathrm{codim}\,(N)$, the collection

$$U_i=p^{-1}\big(\{x_i\neq 0\}\big)\cup\big\{[v]\in\mathbb{P}(\nu_N(M)\big|_{U\cap N}):\mathrm{d}x_i(v)\neq 0\big\}$$

forms an open cover of $p^{-1}(U)$. On each U_i, one defines coordinates $(\tilde{x},y)$ on $\mathrm{Blup}(M,N)$, in which the blowdown map reads (see e.g. [Obs21, Remark 5.29] for details)

$$p(\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_{\mathrm{codim}\,(N)},y)=(\tilde{x}_i\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_i\tilde{x}_{\mathrm{codim}\,(N)},y). \qquad (1.25)$$

Notice that the hyperplane $\mathbb{P}(\nu_N(M))\cap U_i$ is given by $\tilde{x}_i=0$, and that the chart obtained by restriction is the well-known chart on the projective bundle induced by the fibrewise linear coordinates $(\mathrm{d}x,y)$ on the normal bundle $\nu_N(M)$.

Using (1.25) one can show the following Lemma (see [LGLR24, Prop. 1.5.40] for N a point, or Lemma 4.2.1).

Lemma 1.1.9 *A vector field X on M is p-related to some vector field $\tilde{X}$ on* $\mathrm{Blup}(M,N)$ *if and only if X is tangent to N. In that case, $\tilde{X}$ is unique and tangent to* $\mathbb{P}(\nu_N(M))\subseteq\mathrm{Blup}(M,N)$.

Blowups of Lie algebroids

Starting from a Lie algebroid $A\Rightarrow M$ and a closed and embedded Lie subalgebroid $B\Rightarrow N$, the blowup $\mathrm{Blup}(A,B)$ will in general not carry a natural structure of a Lie algebroid. In fact, if $\mathrm{rank}\,(B)<\mathrm{rank}\,(A)$, already the vector bundle projection $\pi\colon A\to M$ of A will not lift to a map $\mathrm{Blup}(A,B)\to\mathrm{Blup}(M,N)$. Instead, one needs to consider the space

$$\mathrm{Blup}_\pi(A,B)=(A\setminus\pi^{-1}(N))\cup\mathbb{P}\Big(\frac{TA\big|_B}{TB}\setminus\frac{(T\pi)^{-1}(TN)\big|_B}{TB}\Big)\subseteq\mathrm{Blup}(A,B).$$

The blowdown map $p\colon\mathrm{Blup}(A,B)\to A$ restricts to a map

$$p_A\colon\mathrm{Blup}_\pi(A,B)\to A,$$

with image given by $A\big|_{M\setminus N}\cup B$. The vector bundle projection $\pi\colon A\to M$ lifts to a map

$$\mathrm{Blup}(\pi)\colon\mathrm{Blup}_\pi(A,B)\to\mathrm{Blup}(M,N). \qquad (1.26)$$

Moreover, (1.26) is a vector bundle ([DS21, Fact 2.15]), and we can easily describe its sections (see e.g. [Obs21, Proposition 5.55]).

Remark 1.1.10 (Sections of $\mathrm{Blup}_\pi(A,B)$) Let $\pi\colon A\to M$ be a vector bundle and $B\to N$ a closed and embedded subbundle. Then

$$\mathrm{Blup}(\pi)\colon\mathrm{Blup}_\pi(A,B)\to\mathrm{Blup}(M,N)$$

is a vector bundle with sections given by the $\mathscr{C}^\infty(\mathrm{Blup}(M,N))$-span of

$$\{\mathrm{Blup}(s)\colon \mathrm{Blup}(M,N) \to \mathrm{Blup}_\pi(A,B) : s \in \Gamma(A,B)\}. \tag{1.27}$$

Here, $\Gamma(A,B)$ denote sections of A that restrict to sections of B over N. When writing $\mathrm{Blup}(s)$ for $s \in \Gamma(A,B)$, we mean a map

$$\mathrm{Blup}(s)\colon \mathrm{Blup}_s(M,N) \to \mathrm{Blup}(A,B)$$

analog to (1.26). One can check, however, that $\mathrm{Blup}_s(M,N) = \mathrm{Blup}(M,N)$ and $\mathrm{im}\,(\mathrm{Blup}(s)) \subseteq \mathrm{Blup}_\pi(A,B)$.

To equip $\mathrm{Blup}_\pi(A,B)$ with the structure of a Lie algebroid, we define anchor and bracket on the set of generators (1.27). Let $s, s' \in \Gamma(A,B)$.

- The anchor $\sharp_{\mathrm{Blup}}$ is defined by

  $$\sharp_{\mathrm{Blup}}(\mathrm{Blup}(a)) = \widetilde{\sharp(a)},$$

  where $\widetilde{\cdot}$ denotes the lift of a vector field according to Lemma 1.1.9.
- The bracket $[\,\cdot\,,\,\cdot\,]_{\mathrm{Blup}}$ is defined by

  $$[\mathrm{Blup}(a), \mathrm{Blup}(b)]_{\mathrm{Blup}} = \mathrm{Blup}([a,b]),$$

  and extended to all sections via the Leibniz rule.

With this structure, $\mathrm{Blup}_\pi(A,B) \Rightarrow \mathrm{Blup}(M,N)$ becomes a Lie algebroid, and the blowdown map $p_A\colon \mathrm{Blup}_\pi(A,B) \to A$ is a Lie algebroid morphism.

Remark 1.1.11 Note that there exists a corresponding construction for Lie groupoids [DS21], see also [Obs21]. The projective blowup construction has been used to desingularise proper groupoids [PTW21, Wan18].

Example 1.1.12 The construction of **Elementary modifications** [GL13, Kla17, Lan21] can be expressed in terms of blowups. If the submanifold $N \subseteq M$ has $\mathrm{codim}\,(N) = 1$, the blowdown map $p\colon \mathrm{Blup}(M,N) \to M$ is a diffeomorphism. In particular, the sections of the Lie algebroid blowup are precisely given by

$$\Gamma(\mathrm{Blup}_\pi(A,B)) = \Gamma(A,B). \tag{1.28}$$

Examples of elementary modifications are

1. the log-tangent bundle $T^b_N M = \mathrm{Blup}_{\pi_{TM}}(TM, TN)$;
2. the scattering tangent bundle $\mathrm{Blup}_{\pi_{T^b_N M}}(T^b_N M, 0_N)$.

1.2 Outline of the chapters

We give an overview over the contents and main results of this thesis. Chapter 2 and 3 develop tools to compute Lie algebroid cohomology, using spectral sequences and real projective blowups, respectively. Chapter 4 and 5 are concerned with the question of lifting geometric structures (twisted Dirac and Poisson, respectively) to the real projective blowup. Finally, in Chapter 6 we study the relationship between the singular foliations of Lie algebroids and Dirac structures and their pullback via a transverse map. Moreover, we discuss the singular foliation of certain blowups of Lie algebroids.

1.2.1 Chapter 2

Chapter 2 contains [MS24], which is available on arXiv under arXiv:2405.00419.

Lie algebroid cohomology encodes many important geometric properties of the underlying Lie algebroid, such as invariant functions, infinitesimal automorphisms, and deformations. However, Lie algebroid cohomology is, in general, quite difficult to compute. Even if the base manifold is compact, Lie algebroid cohomology can be infinite dimensional. Points on M in which the anchor map drops rank pose an additional difficulty. One tool to compute cohomologies is given by spectral sequences. In this chapter, we define and show convergence properties of the Serre spectral sequence of Lie algebroids induced by the inclusion of a Lie subalgebroid. We briefly outline the construction of spectral sequences.

Spectral sequences

Let

$$\ldots \subseteq \mathcal{F}^p\Omega^\bullet(A) \subseteq \mathcal{F}^{p-1}\Omega^\bullet(A) \subseteq \ldots \subseteq \mathcal{F}^0\Omega^\bullet(A) = \Omega^\bullet(A)$$

be a differential graded filtration on $\Omega^\bullet(A)$, i.e. a sequence of subspaces, each stable under d. Then, in a first step, one can consider the induced map

$$\mathrm{d}_0\colon E_0^{p,q} := \frac{\mathcal{F}^p\Omega^{p+q}(A)}{\mathcal{F}^{p+1}\Omega^{p+q}(A)} \to \frac{\mathcal{F}^p\Omega^{p+q+1}(A)}{\mathcal{F}^{p+1}\Omega^{p+q+1}(A)} =: E_0^{p,q+1}.$$

Elements in the kernel of d_0 are not necessarily mapped to zero under d, but instead lie further down in the filtration. The collection of maps $\mathrm{d}_0\colon E_0^{p,\bullet} \to E_0^{p,\bullet+1}$ again squares to zero, and d induces

$$\mathrm{d}_1\colon E_1^{p,q} := \mathrm{H}^{p,q}(E_0, \mathrm{d}_0) \to \mathrm{H}^{p+1,q}(E_0, \mathrm{d}_0) =: E_1^{p+1,q}.$$

One can then iterate the process, and $\{E_r^{\bullet,\bullet}, \mathrm{d}_r\}_{r\geq 0}$ is called the spectral sequence associated to the filtration. We say that the spectral sequence converges to a space $X^\bullet$ if the notion of $E_\infty^{\bullet,\bullet}$ makes sense and

$$X^n \simeq \bigoplus_{i+j=n} E_\infty^{i,j}.$$

Several spectral sequences for Lie algebroids are known in the literature.

Example 1.2.1

1. The Hochschild-Serre spectral sequence [HS53] associated to a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$;
2. The spectral sequence of a Lie algebroid extension
$$0 \to L \to A \to B \to 0$$
([Mac05], see also [Bra10]);
3. The Leray-Serre spectral sequence [Ser51] for the de Rham cohomology of a locally trivial fibre bundle $\pi\colon M \to Q$ (developed in [Hat60]);
4. The spectral sequence of a regular Poisson manifold [Vai90].

Main results

The Serre spectral sequence is constructed as follows. Given a Lie subalgebroid $L \Rightarrow N$ of $A \Rightarrow M$, the pullback along the inclusion $i\colon L \hookrightarrow A$ yields a differential graded ideal in $\Omega^\bullet(A)$,

$$\mathcal{I} := \ker\big(i^*\colon \Omega^\bullet(A) \to \Omega^\bullet(L)\big).$$

This ideal gives rise to a descending differential graded filtration on $\Omega^\bullet(A,V)$ by setting

$$\mathcal{F}^p\Omega^\bullet(A,V) := \big(\wedge^p \mathcal{I} \wedge \Omega(A,V)\big)^\bullet \subseteq \Omega^\bullet(A,V).$$

This filtration induces a spectral sequence $\{E_r^{p,q}\}_{r\geq 0}$ which we call the **Serre spectral sequence** of the Lie subalgebroid $i\colon L \hookrightarrow A$.

Regarding convergence, we obtain the following result in Section 2.2.1.

Theorem 1.2.2 *Let $L \Rightarrow N$ be a Lie subalgebroid and $V \to M$ a representation of $A \Rightarrow M$.*

1. *If $N = M$, i.e. L is a wide subalgebroid, then the Serre spectral sequence associated to L converges to the cohomology of A with values in V.*
2. *If $N \subseteq M$ is a closed and embedded submanifold of positive codimension, then the Serre spectral sequence converges to the formal cohomology of A along N with values in V.*

Using the notion of Lie algebroid representations up to homotopy introduced in [AC13], we can describe the first page of the Serre spectral sequence in the settings of Theorem 1.2.2 in full generality.

Theorem 1.2.3 *Let $L \Rightarrow N$ be a Lie subalgebroid and $V \to M$ a representation of $A \Rightarrow M$, where $N \subseteq M$ is a closed, embedded submanifold. The conormal bundle of L in A is canonically a VB-algebroid $\nu_L(A)^* \Rightarrow L^{\mathrm{ann}}$, which corresponds to a representation up to homotopy of L on $\nu_N(M)^* \oplus L^{\mathrm{ann}}$. The first page of the Serre spectral sequence of L is isomorphic to the cohomology of L with coefficients in representations up to homotopy, as follows*

$$E_1^{p,q} \simeq \mathrm{H}(L, \wedge^p(\nu_N(M)^* \oplus L^{\mathrm{ann}}) \otimes V\big|_N)^q. \tag{1.29}$$

These representations up to homotopy are classical representations in the following cases.

1. *If $N = M$, i.e. L is a wide Lie subalgebroid, then* (1.29) *reduces to*

   $$E_1^{p,q} \simeq \mathrm{H}^q(L, \wedge^p L^{\mathrm{ann}} \otimes V),$$

   where the representation of L on $L^{\mathrm{ann}} \simeq (A/L)^*$ is the dual Bott connection.
2. *If N is an invariant submanifold of A and $L = A\big|_N$, then* (1.29) *reduces to*

   $$E_1^{p,q} \simeq \mathrm{H}^{p+q}(A\big|_N, S^p\nu_N(M)^* \otimes V\big|_N).$$

We study the case of N a submanifold of positive codimension further in Section 2.2.3 for N an invariant submanifold, a transversal, or more generally, a coregular submanifold.

For suitable choices of wide Lie subalgebroids one recovers all spectral sequences for Lie algebroids listed in Example 1.2.1.

If $A = \pi^! B$ is the pullback Lie algebroid along a locally trivial fibration $\pi\colon M \to Q$ and $L = \ker T\pi$, we obtain the following generalisation of the Leray-Serre spectral sequence for de Rham cohomology.

Theorem 1.2.4 *Let $B \Rightarrow Q$ be a Lie algebroid over the base of a locally trivial fibre bundle $\pi\colon M \to Q$ with typical fibre F. If $\mathrm{H}^\bullet(F)$ is finite dimensional, the associated spectral sequence computing $\mathrm{H}^\bullet(\pi^! B)$ satisfies*

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \mathcal{H}^q(\ker T\pi)),$$

where the representation of B on $\mathcal{H}^q(\ker T\pi)$ is the pullback via the anchor $\sharp\colon B \to TN$ of the Gauss-Manin connection, i.e. $\nabla_b = \nabla^{GM}_{\sharp b}$. Here, the bundle $\mathcal{H}^q(\ker T\pi) \to Q$ is the finite dimensional vector bundle with fibres $\mathcal{H}^q(\ker T\pi)_x = \mathrm{H}^q(\pi^{-1}(x))$.

Another class of examples are given by submersions by Lie algebroids, discussed in Section 2.3.3. Submersions by Lie algebroids, introduced and studied in [Fre19], consist of a Lie algebroid $A \Rightarrow M$ and a surjective submersion $\pi\colon M \to Q$, such that $T\pi \circ \sharp\colon A \to TQ$ is pointwise surjective. A spectral sequence for the cohomology of a submersion by Lie algebroids was developed in [Fre19]. We compare this to the Serre spectral sequence of the Lie subalgebroid $L := \ker(T\pi \circ \sharp)$. Moreover, we prove the following result.

Theorem 1.2.5 *Let $(A \Rightarrow M, \pi\colon M \to Q)$ be a locally trivial submersion by Lie algebroids and $V \to M$ a representation of A. The spaces on the second page of the Serre spectral sequence associated to the Lie subalgebroid $L = \ker(T\pi \circ \sharp)$ are isomorphic to the sheaf cohomology of Q,*

$$E_2^{p,q} \simeq \mathrm{H}^p(Q, \mathcal{S}^q_{L,V}),$$

where $\mathcal{S}^q_{L,V}$ is a locally constant sheaf, and it sends an open subset $U \subseteq Q$ to

$$\mathcal{S}^q_{L,V}(U) = \big\{c \in \mathrm{H}^q(L\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) : \nabla c = 0\big\}.$$

Here, ∇ is a generalised representation of TQ on the $\mathscr{C}^\infty(Q)$-module $\mathrm{H}^q(L, V)$.

We then apply Theorem 1.2.5 to horizontally nondegenerate Dirac structures (see [Wad08], and [Vor01, Vor05] in the Poisson case). We obtain descriptions of the cohomology of such Dirac structures in low degrees; in particular, we reproduce and generalise results for horizontally nondegenerate Poisson structures obtained in [VBV18].

In Section 2.4 we consider extensions $0 \to L \to A \to B \to 0$ for which the Lie subalgebroid L is abelian. Then we can describe the differential on E_2 by means of the extension class

$$[\gamma] \in \mathrm{H}^2(B, L).$$

Note that, if the base map of $A \to B$ is not given by id_M, we make use of the notion of a generalised representation. Namely, we obtain the following generalisation of [Mac05, Theorem 7.4.11] (for $B = TQ$), [HS53, Theorem 8] (for Lie algebras), and [MZ22, Corollary 4.3] (for the anchor of a regular Lie algebroid).

Theorem 1.2.6 *Let* $0 \to L \to A \to B \to 0$ *be an abelian extension, and let* V *be a representation of* A *on which* L *acts trivially. The second page of the Serre spectral sequence,* $\mathrm{d}_2\colon E_2^{p,q} \to E_2^{p+2,q-1}$*, can be identified as*

$$\big((-1)^p \mathrm{i}_{[\gamma]}\colon \mathrm{H}^p(B, \Omega^q(L, V)) \to \mathrm{H}^{p+2}(B, \Omega^{q-1}(L, V))\big).$$

1.2.2 Chapter 3

Chapter 3 contains [Sch24], which is available on arXiv under arXiv:2406.17668.

Another tool we develop for the computation of Lie algebroid cohomology uses blowups. Recall that the choice of a closed and embedded Lie subalgebroid $B \Rightarrow N$ of a Lie algebroid $A \Rightarrow M$ gives rise to a new Lie algebroid, $\mathrm{Blup}(A, B)$ (note that we omit the subscript in the blowup of Lie algebroids). Part of the construction is the existence of a blowdown map

$$\begin{tikzcd}
\mathrm{Blup}(A, B) \arrow[r, "p_A"] \arrow[d, Rightarrow] & A \arrow[d, Rightarrow] \\
\mathrm{Blup}(M, N) \arrow[r, "p"] & M
\end{tikzcd}$$

The blowdown map is morphism of Lie algebroids and, therefore, induces a map in cohomology

$$p_A^*\colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(\mathrm{Blup}(A, B)).$$

The idea is to study this blowdown map in cohomology to obtain insight to Lie algebroid cohomologies. For this, two scenarios are possible. Either the blown-up Lie algebroid is of interest and we would like to understand its cohomology in relation to $\mathrm{H}^\bullet(A)$, or we are interested in computing $\mathrm{H}^\bullet(A)$. In the second case, one chooses the Lie subalgebroid B in such a way that $\mathrm{H}^\bullet(\mathrm{Blup}(A, B))$ is easier to compute, e.g. $\mathrm{Blup}(A, B)$ is a regular Lie algebroid, and then draws conclusions regarding $\mathrm{H}^\bullet(A)$ via the blowdown map.

Main results

The blowdown map $p_A\colon \mathrm{Blup}(A, B) \to M$ induces a short exact sequence

$$0 \to \Omega^\bullet(A) \xrightarrow{p_A^*} \Omega^\bullet(\mathrm{Blup}(A, B)) \to \frac{\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\Omega^\bullet(A)} \to 0.$$

This short exact sequence induces a long exact sequence in cohomology, given by

$$\cdots \to \mathrm{H}^\bullet(A) \xrightarrow{p_A^*} \mathrm{H}^\bullet(\mathrm{Blup}(A, B)) \to \mathrm{H}^\bullet\Big(\frac{\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\Omega^\bullet(A)}\Big) \to \mathrm{H}^{\bullet+1}(A) \to \dots$$

Studying the blowdown map $p_A^*\colon \Omega^\bullet(A) \to \Omega^\bullet(\mathrm{Blup}(A,B))$ we see that there is a canonical isomorphism

$$\frac{\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\Omega^\bullet(A)} \simeq \frac{\mathscr{I}_{\mathbb{P}}^\infty\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\mathscr{I}_N^\infty\Omega^\bullet(A)}, \tag{1.30}$$

where $\mathscr{I}_N^\infty$ denote infinity jets along N.

We then use (1.30) to compute the cohomology of the blowup of a transversal $\iota\colon N \hookrightarrow M$.

Theorem 1.2.7 (The blowup of transversals) *Let $\iota\colon N \hookrightarrow M$ be a closed transversal of $A \Rightarrow M$ and denote the projection of the projective bundle $\mathbb{P}(\nu_N(M)) \subseteq \mathrm{Blup}(M,N)$ by $\pi_{\mathbb{P}}\colon \mathbb{P}(\nu_N(M)) \to N$. Let $\iota^!A \Rightarrow N$ be the pullback of A to N.*

1. *If* codim N *is odd, we have an isomorphism*

$$\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)) \simeq \mathrm{H}^\bullet(A) \oplus \mathrm{H}^{\bullet-1}(\iota^!A) \tag{1.31}$$

and, under (1.31), *p_A^* becomes the isomorphism $p_A^*\colon \mathrm{H}^\bullet(A) \xrightarrow{\simeq} \mathrm{H}^\bullet(A) \oplus 0$.*

2. *If* codim N *is even, there exists a tubular neighbourhood $E \to N$ such that* $\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A))$ *fits into a long exact sequence*

$$\begin{aligned} \ldots \to \mathrm{H}^\bullet(A) \xrightarrow{p_A^*} \mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)) \to \\ \to \mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A\big|_E) \oplus \mathrm{H}^{\bullet-1}(\pi_{\mathbb{P}}^!A) \xrightarrow{g} \mathrm{H}^{\bullet+1}(A) \to \ldots \end{aligned}$$

where $g = i \circ \mathrm{pr}_{\mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A|_E)}$. Here, by $\mathrm{H}_{\mathrm{cv}}^\bullet(A\big|_E)$ we denote compact vertical cohomology and by $i\colon \mathrm{H}_{\mathrm{cv}}^\bullet(A\big|_E) \to \mathrm{H}^\bullet(A)$ the natural map.

We also compute the de Rham cohomology of a real projective blowup using similar methods.

Theorem 1.2.7 can be seen as a generalisation of the result obtained by Mazzeo-Melrose on b-cohomology ([GMP14, MT14], see [Mel93] for the original result). Indeed, recall that by [GL13, Section 2.4.1] we can write the b-tangent bundle associated to a closed hypersurface $N \subseteq M$ by

$$T_N^bM = \mathrm{Blup}(TM, TN),$$

which is a blowup of a codimension 1 transversal. Theorem 1.2.7 then reproduces the Mazzeo-Melrose decomposition for b-cohomology. In this sense, Theorem 1.2.7 can be seen as a generalisation of Mazzeo-Melrose as it allows for arbitrary Lie algebroids and transversals of arbitrary codimension.

To show the second part of Theorem 1.2.7, we develop a Gysin-like long exact sequence for the cohomology of the pullback of a Lie algebroid to a sphere bundle, making use of the Serre spectral sequence.

Theorem 1.2.8 (Gysin sequence for Lie algebroids) *Let $B \Rightarrow N$ be a Lie algebroid with anchor $\sharp$, $\pi\colon \mathbb{S} \to N$ a sphere bundle of rank k with orientation* *bundle* $o(\mathbb{S}) \to N$*, and* $\pi^! B \Rightarrow \mathbb{S}$ *the pullback Lie algebroid. There exists a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^\bullet(B) \xrightarrow{(\pi^!)^*} \mathrm{H}^\bullet(\pi^! B) \xrightarrow{(\pi^!)_*} \mathrm{H}^{\bullet-k}(B, o(\mathbb{S})) \xrightarrow{\wedge\sharp^* e} \mathrm{H}^{\bullet+1}(B) \longrightarrow \ldots \tag{1.32}$$

Here, $(\pi^!)_*$ *denotes fibre integration and* $e \in \mathrm{H}^{k+1}(N, o(\mathbb{S}))$ *is the Euler class of the sphere bundle.*

If $N \subseteq M$ is an invariant submanifold and $B = A\big|_N$, we show that the pullback by the blowdown map induces a map between Serre spectral sequences associated to the inclusions $A\big|_N \hookrightarrow A$ and $\mathrm{Blup}(A, A\big|_N)\big|_{\mathbb{P}} \hookrightarrow \mathrm{Blup}(A, A\big|_N)$.

As an example, we consider the action Lie algebroid $A = \mathfrak{so}(3) \ltimes \mathbb{R}^3$. Blowing up the fibre over the origin leads to a regular Lie algebroid, whose cohomology we compute. We then make use of the above comment to show that in this case, the cohomology of the quotient complex

$$\frac{\mathscr{I}_{\mathbb{P}}^{\infty}\Omega^\bullet(\mathrm{Blup}(A, A\big|_{\{0\}}))}{p_A^* \mathscr{I}_N^{\infty}\Omega^\bullet(A)}$$

vanishes. Then the isomorphism (1.30) reproduces the well-known cohomology of $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ [GW92].

Finally, using the Lie algebroid $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^3$ we show that one cannot always obtain a regular Lie algebroid by blowing up orbits.

1.2.3 Chapter 4

This chapter contains [MSZ25].

Given a geometric structure on a manifold M and a closed and embedded submanifold $N \subseteq M$, one can ask, under which conditions the structure "lifts" to the blowup in a suitable sense. In [Pol97], Polishchuk gave a complete answer to this question for Poisson structures in the context of algebraic varieties. We state the result here in its differential-geometric form as given in [GL13].

Theorem 1.2.9 (Polishchuk [Pol97]) *Let* (M, π) *be a Poisson manifold and* $N \subseteq M$ *a closed, embedded submanifold of codimension* > 1*. Then the following are equivalent.*

(a) There exists a Poisson structure on $\mathrm{Blup}(M, N)$ *such that the blowdown map* $p\colon \mathrm{Blup}(M, N) \to M$ *is Poisson;*

(b) The submanifold N *is Poisson and for all* $y \in N$ *the Lie algebra structure on the conormal space* $(T_y N)^{\mathrm{ann}}$*, where* $\cdot^{\mathrm{ann}}$ *denotes the annihilator, is either abelian or* $\mathbb{R} \ltimes \mathbb{R}^n$*, the semidirect product by the diagonal representation* $\lambda \mapsto \lambda\mathrm{id}$ *of* $\mathbb{R}$*.*

A possible generalisation of Poisson structures is given by (twisted) Dirac structures. In this chapter, we fully answer the following question. Given an H-twisted Dirac structure on a manifold M and a closed and embedded submanifold $N \subseteq M$, under which conditions does there exist a (necessarily unique and p^*H-twisted) Dirac structure on $\mathrm{Blup}(M, N)$, which extends the original Dirac structure from $M \setminus N$ and, thus, lifts the Dirac structure to the blowup? And in that case, is the blowdown map forward or backward Dirac?

Main results

Regarding lifting of Dirac structures we obtain the following result.

Theorem 1.2.10 *Let D be a twisted Dirac structure on a manifold M and $N \subseteq M$ a connected, closed, and embedded submanifold of codimension > 1. Then D lifts to a twisted Dirac structure on* $\mathrm{Blup}(M, N)$ *if and only if one of the following conditions holds.*

(1) $N \subseteq M$ is a transversal.

(2) $N \subseteq M$ is an invariant submanifold and the bundle $(TN)^{\mathrm{ann}} \to N$ of Lie algebras (see (1.21)*) satisfies one of the following conditions:*

  (a) Each fibre is either abelian or isomorphic to the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^n$ by the diagonal representation of $\mathbb{R}$ on $\mathbb{R}^n$.

  (b) The fibres are all isomorphic to $\mathfrak{so}(3)$.

The blowdown map is a backward Dirac map in case (1), and it is a forward Dirac map in case (2).

Theorem 1.2.10 is surprising in the sense that, given any connected submanifold N, the existence of a lift of D implies the restrictive condition that N is either invariant or a transversal. Moreover, the list of possible conormal Lie algebra structures in the invariant case contains only one additional Lie algebra (namely $\mathfrak{so}(3)$) compared to Polishchuk's result (see Theorem 1.2.9).

The Lie algebras appearing in Theorem 1.2.10 are characterised by the following. Given a Lie algebra $\mathfrak{g}$, the **height** of $\xi \in \mathfrak{g}^* \setminus \{0\}$ is defined to be the integer $k \in \mathbb{N}_0$ such that

$$\xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^k \neq 0 \quad \text{and} \quad \xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} = 0.$$

We say that $\mathfrak{g}$ is of **constant height** k if all $\xi \in \mathfrak{g}^* \setminus \{0\}$ have the same height k.

While proving Theorem 1.2.10, in Section 4.6 we show that, in case N is invariant, the Dirac structure lifts if and only if the conormal Lie algebras are of the same constant height for every point in N. Hence, to complete the proof of Theorem 1.2.10, we use structure theory of semisimple Lie algebras to classify all real Lie algebras of constant height.

Theorem 1.2.11 *Any real Lie algebra $\mathfrak{g}$ of constant height is isomorphic to one of the following.*

- *An abelian Lie algebra $\mathbb{R}^n$—this has height* 0.
- *The semi-direct product $\mathbb{R} \ltimes \mathbb{R}^n$, for the representation $\lambda \mapsto \lambda \mathrm{id}_{\mathbb{R}^n}$—this has height* 0.
- *The Lie algebra $\mathfrak{so}(3)$—this has height* 1.

1.2.4 Chapter 5

This chapter contains [Sch25], in which we give a detailed and elementary proof for Theorem 1.2.9. Moreover, in [GL13] the condition on the conormal Lie algebras was expressed solely in terms of the linearisation π_{lin} of the Poisson bivector field along N, that is

$$\pi_{\mathrm{lin}} = \xi \wedge v^{\mathrm{ver}} \in \Gamma(\wedge^2 T\nu_N(M)). \tag{1.33}$$

Here, ξ denotes the Euler vector field on the normal bundle $\nu_N(M) \to N$, and v^{ver} is the vertical lift ("constant extension") of a section $v \in \nu_N(M)$ to a vector field on $\nu_N(M)$. In Section 5.5 we give an interpretation of the specific form (1.33).

1.2.5 Chapter 6

This chapter contains [SZ25].

We study singular foliations in relation with pullbacks and blowups. A singular foliation is a $\mathscr{C}^\infty(M)$-submodule of $\Gamma_{\mathrm{c}}(TM)$, the compactly supported vector fields, that is closed under the Lie bracket [AS09]. Both Lie algebroids $A \Rightarrow M$ and Dirac structures $D \subseteq \mathbb{T}M$ induce a singular foliation on M via $\sharp(\Gamma_{\mathrm{c}}(A))$ and $\mathrm{pr}_{TM}(\Gamma_{\mathrm{c}}(D))$, respectively. A singular foliation in this sense induces, but is not equivalent to, a partition of M into submanifolds of varying dimensions.

Main results

In the first part of this chapter, we consider a Dirac or a Lie algebroid structure on a manifold M and their underlying singular foliations. Given a smooth map transverse to the structure, we give proofs that the following operations commute with taking pullbacks:

- taking the singular foliation of a Lie algebroid (see Proposition 6.3.1),
- taking the Lie algebroid underlying a Dirac structure (see Proposition 6.4.1).

In Section 6.5 we consider a submanifold $\iota\colon N \hookrightarrow M$ transverse to the Dirac or Lie algebroid structure. In this case, the blowdown map $p\colon \mathrm{Blup}(M,N) \to N$ is also transverse, and by the above results the singular foliations underlying the pullback Dirac and pullback Lie algebroid structure coincide. However, in this case we can also consider the Lie algebroid blowup of any Lie subalgebroid $L \Rightarrow N$ to obtain a Lie algebroid structure on $\mathrm{Blup}(M,N)$, which is not isomorphic to the pullback Lie algebroid. We compute the singular foliation of $\mathrm{Blup}(A,L)$, where $L \Rightarrow N$ is any Lie subalgebroid with $\ker \sharp|_{A|_N} \subseteq L$.

Theorem 1.2.12 *Let $L \Rightarrow N$ be a Lie subalgebroid of $A \Rightarrow M$, supported over a transversal $N \subseteq M$, such that* $\ker \sharp|_{A|_N} \subseteq L$*. Then for the singular foliation $\mathcal{F}_{\mathrm{Blup}}$ of* $\mathrm{Blup}(A,L)$*, we obtain*

$$\mathcal{F}_{\mathrm{Blup}} = \mathrm{span}_{\mathscr{C}^\infty_{\mathrm{c}}(\mathrm{Blup}(M,N))}\{\tilde{Y} : Y \in \mathcal{F}_{A,L}\}.$$

Here, $\mathcal{F}_{A,L} := \sharp(\Gamma_c(A,L))$*, and* $\tilde{\cdot}$ *denotes the lift of a vector field tangent to* N *to the blowup, see Lemma 1.1.9.*

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Chapter 2

The Serre-Leray spectral sequence of Lie algebroids

This chapter contains [MS24], submitted, with the following changes. [MS24, Section 2], which contains all mathematical background regarding Lie algebroids needed for this chapter, has been moved to the introduction as Section 1.1.2. Moreover, Corollary 2.4.2 has been moved to this chapter from [Sch24, Corollary 6.5]. I did not contribute to the discussion on coupling Poisson Dirac structures and normal forms around presymplectic leaves in Section 2.3.3, as well as Vaisman's spectral sequence in Section 2.4.1.

Abstract

We study a spectral sequence approximating Lie algebroid cohomology associated to a Lie subalgebroid. This is a simultaneous generalisation of several classical constructions in differential geometry, including the Leray–Serre spectral sequence for de Rham cohomology associated to a fibration [Ser51], the Hochschild-Serre spectral sequence for Lie algebras [HS53], and the Mackenzie spectral sequence for Lie algebroid extensions [Mac05]. We show that, for wide Lie subalgebroids, the spectral sequence converges to the Lie algebroid cohomology, and that, for Lie subalgebroids over proper submanifolds, the spectral sequence converges to the formal Lie algebroid cohomology. We discuss applications and recover several constructions in Poisson geometry in which this spectral sequence has appeared naturally in the literature.

2.1 Introduction

Lie algebroids generalise simultaneously various geometric structures, including Lie algebras, manifolds (via their tangent bundles), foliations, Lie algebra actions, Poisson structures, (generalised) complex structures (in the setting of complex Lie algebroids) etc.

To a Lie algebroid $A \Rightarrow M$ (with anchor denoted by $\sharp\colon A \to TM$) one associates the cochain complex of differential forms on A,

$$(\Omega^\bullet(A), \wedge, \mathrm{d}_A).$$

This assignment is a fully faithful contravariant functor from the category of Lie algebroids to that of differential graded commutative algebras. The resulting cohomology groups, $\mathrm{H}^\bullet(A)$, form the Lie algebroid cohomology of A. Given a representation $V \to M$ of A, forms on A with values in V form a differential graded $\Omega^\bullet(A)$-module, denoted by $(\Omega^\bullet(A, V), \mathrm{d}_A)$, which yields the Lie algebroid cohomology of A with values in V, denoted by $\mathrm{H}^\bullet(A, V)$.

Lie algebroid cohomology encodes important geometric information (e.g. invariant functions, infinitesimal automorphisms of Poisson and other geometric structures, deformations, etc.). However, calculating it can be quite difficult in general. One of the issues is that the defining complex is typically non-elliptic and so, even over a compact manifold, the cohomology groups might be infinite dimensional. Another difficulty is to understand and control the behaviour around points where the anchor drops rank. Moreover, there are few computational tools available in general. In this paper, we build such a tool, namely a spectral sequence approximating Lie algebroid cohomology associated to a Lie subalgebroid, which simultaneously generalises several classical constructions.

The spectral sequence is constructed as follows. Given a Lie subalgebroid $L \Rightarrow N$ of $A \Rightarrow M$, the pullback along the inclusion $i\colon L \hookrightarrow A$ yields a differential graded ideal in $\Omega^\bullet(A)$,

$$\mathcal{I} := \ker\big(i^* : \Omega^\bullet(A) \to \Omega^\bullet(L)\big).$$

This ideal gives rise to a descending, differential graded filtration on $\Omega^\bullet(A, V)$ by setting

$$\mathcal{F}^p\Omega^\bullet(A, V) := \big(\wedge^p \mathcal{I} \wedge \Omega(A, V)\big)^\bullet \subseteq \Omega^\bullet(A, V).$$

This filtration induces a spectral sequence $\{E_r^{p,q}\}_{r\geq 0}$ which we call the **Serre spectral sequence** of the Lie subalgebroid $i\colon L \hookrightarrow A$. This construction reproduces important spectral sequences known in the literature via suitable choices of the Lie subalgebroid.

1. For a Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, we recover the classical Hochschild-Serre spectral sequence [HS53].

2. The spectral sequence of a Lie algebroid extension ([Mac05], see also [Bra10])
$$0 \to L \to A \to B \to 0$$
is the Serre spectral sequence of $L \hookrightarrow A$ (Section 2.3).

3. The Leray-Serre spectral sequence [Ser51] for de Rham cohomology of a locally trivial fibre bundle $\pi\colon M \to Q$ (developed in [Hat60]) is the Serre spectral sequence of the vertical distribution, i.e. of the subbundle $\ker T\pi \subseteq TM$ (Section 2.3.1). The construction works similarly for the pullback of a Lie algebroid along a fibration (Section 2.3.2).

4. The spectral sequence of a regular Poisson manifold [Vai90] is the Serre spectral sequence of the kernel of the anchor map (Section 2.4.1).

Regarding convergence, we obtain the following result in Section 2.2.1.

Theorem 2.1.1 *Let $L \Rightarrow N$ be a Lie subalgebroid and $V \to M$ a representation of $A \Rightarrow M$.*

1. *If $N = M$, i.e. L is a wide subalgebroid, then the Serre spectral sequence associated to L converges to the cohomology of A with values in V.*
2. *If $N \subseteq M$ is a closed and embedded submanifold of positive codimension, then the Serre spectral sequence converges to the formal cohomology of A along N with values in V.*

Using the notion of Lie algebroid representations up to homotopy introduced in [AC13], we can describe the first page of the Serre spectral sequence in the settings of Theorem 2.1.1.

Theorem 2.1.2 *Let $L \Rightarrow N$ be a Lie subalgebroid and $V \to M$ a representation of $A \Rightarrow M$, where $N \subseteq M$ is a closed, embedded submanifold. The conormal bundle of L in A is canonically a VB-algebroid $\nu_L(A)^* \Rightarrow L^{\mathrm{ann}}$, which corresponds to a representation up to homotopy of L on $\nu_N(M)^* \oplus L^{\mathrm{ann}}$. The first page of the Serre spectral sequence of L is isomorphic to the cohomology of L with coefficients in representations up to homotopy, as follows*

$$E_1^{p,q} \simeq \mathrm{H}(L, \wedge^p(\nu_N(M)^* \oplus L^{\mathrm{ann}}) \otimes V\big|_N)^q. \tag{2.1}$$

These representations up to homotopy are classical representations in the following cases.

1. *If $N = M$, i.e. L is a wide Lie subalgebroid, then* (2.1) *reduces to*
$$E_1^{p,q} \simeq \mathrm{H}^q(L, \wedge^p L^{\mathrm{ann}} \otimes V),$$
where the representation of L on $L^{\mathrm{ann}} \simeq (A/L)^*$ is the dual Bott connection.
2. *If N is an invariant submanifold of A and $L = A\big|_N$, then* (2.1) *reduces to*
$$E_1^{p,q} \simeq \mathrm{H}^{p+q}(A\big|_N, S^p\nu_N(M)^* \otimes V\big|_N).$$

From Section 2.3 on, we will consider subalgebroids which fit into a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, Rightarrow] & A \arrow[r] \arrow[d, Rightarrow] & B \arrow[r] \arrow[d, Rightarrow] & 0 \\
 & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q &
\end{tikzcd} \tag{2.2}$$

where the base map $\pi \colon M \to Q$ is a surjective submersion. In this case we can describe the differential on E_1 more explicitly, and make rigorous the interpretation of the second page given in [Bra10], where the Serre spectral sequence of general extensions (2.2) was first considered.

Theorem 2.1.3 *The* $\mathscr{C}^\infty(Q)$*-module* $\mathrm{H}^\bullet(L,V)$ *is a generalised* B*-representation (defined in Section 1.1.2), and the Serre spectral sequence associated to* L *satisfies*

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \mathrm{H}^q(L,V)).$$

In some cases, the generalised representation of B is actually a classical one. For example, if L is abelian and $\pi = \mathrm{id}_M$, the result was already obtained in [Mac05]. If $A = \pi^! B$ is the pullback Lie algebroid along a locally trivial fibration $\pi\colon M \to Q$ and $L = \ker T\pi$, we obtain the following generalisation of the Leray-Serre spectral sequence for de Rham cohomology.

Theorem 2.1.4 *Let* $B \Rightarrow Q$ *be a Lie algebroid over the base of a locally trivial fibre bundle* $\pi\colon M \to Q$ *with typical fibre* F*. If* $\mathrm{H}^\bullet(F)$ *is finite dimensional, the associated spectral sequence computing* $\mathrm{H}^\bullet(\pi^! B)$ *satisfies*

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \mathcal{H}^q(\ker T\pi)),$$

where the representation of B *on* $\mathcal{H}^q(\ker T\pi)$ *is the pullback via the anchor* $\sharp\colon B \to TN$ *of the Gauss-Manin connection, i.e.* $\nabla_b = \nabla^{GM}_{\sharp b}$*. Here, the bundle* $\mathcal{H}^q(\ker T\pi) \to Q$ *is the finite dimensional vector bundle with fibres* $\mathcal{H}^q(\ker T\pi)_x = \mathrm{H}^q(\pi^{-1}(x))$*.*

However, interpreting the generalised representation as a classical one is not always possible. A class of such examples are given by submersions by Lie algebroids, discussed in Section 2.3.3. Submersions by Lie algebroids, introduced and studied in [Fre19], consist of a Lie algebroid $A \Rightarrow M$ and a surjective submersion $\pi\colon M \to Q$, such that $T\pi \circ \sharp\colon A \to TQ$ is pointwise surjective. A spectral sequence for the cohomology of a submersion by Lie algebroids was developed in [Fre19]. We compare this to the Serre spectral sequence of the Lie subalgebroid $L := \ker(T\pi \circ \sharp)$. Moreover, we prove the following result.

Theorem 2.1.5 *Let* $(A \Rightarrow M, \pi\colon M \to Q)$ *be a locally trivial submersion by Lie algebroids and* $V \to M$ *a representation of* A*. The spaces on the second page of the Serre spectral sequence of the Lie subalgebroid* $L = \ker(T\pi \circ \sharp)$ *are isomorphic to the sheaf cohomology of* Q*,*

$$E_2^{p,q} \simeq \mathrm{H}^p(Q, \mathcal{S}^q_{L,V}),$$

where $\mathcal{S}^q_{L,V}$ *is a locally constant sheaf, and it sends an open subset* $U \subseteq Q$ *to*

$$\mathcal{S}^q_{L,V}(U) = \left\{c \in \mathrm{H}^q(L\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) : \nabla c = 0\right\},$$

where ∇ *is a generalised representation of* TQ *on the* $\mathscr{C}^\infty(Q)$*-module* $\mathrm{H}^q(L,V)$*.*

In Section 2.3.3 and 2.3.3 we apply Theorem 2.1.5 to horizontally nondegenerate Dirac structures (see [Wad08], and [Vor01, Vor05] in the Poisson case). We obtain descriptions of the cohomology of such Dirac structures in low degrees; in particular, we reproduce and generalise results for horizontally nondegenerate Poisson structures obtained in [VBV18].

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2.2 The Serre spectral sequence of a Lie subalgebroid

In this section we introduce the Serre spectral sequence of a Lie subalgebroid. After discussing the general construction and convergence properties of the spectral sequence, we describe the zeroth page and the spaces on the first page first for Lie subalgebroids over the whole base manifold and then for Lie subalgebroids over closed and embedded submanifolds.

2.2.1 The spectral sequence

We will recall the basic construction of a spectral sequence associated to a filtered complex (see for example [Wei94, Chapter 5.4], [McC01, Chapter 1.1] or [Mac05, Chapter 7.4]).

On the cochain complex $(\Omega^\bullet(A,V), \mathrm{d}_A)$ consider a differential graded **filtration**,

$$\ldots \subseteq \mathcal{F}^p\Omega^\bullet(A,V) \subseteq \mathcal{F}^{p-1}\Omega^\bullet(A,V) \subseteq \ldots \subseteq \mathcal{F}^0\Omega^\bullet(A,V) = \Omega^\bullet(A,V).$$

The **spectral sequence** associated to this filtration is defined as follows. For $r \geq 0$, let

$$Z_r^{p,q} := \{\omega \in \mathcal{F}^p\Omega^{p+q}(A,V) : \mathrm{d}_A\omega \in \mathcal{F}^{p+r}\Omega^{p+q+1}(A,V)\},$$

where, for $p \leq 0$, $\mathcal{F}^p\Omega^\bullet(A,V) := \Omega^\bullet(A,V)$. Since $Z_{r-1}^{p+1,q-1} \subseteq Z_r^{p,q}$, we can define the quotients

$$E_r^{p,q} := \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + \mathrm{d}_A\, Z_{r-1}^{p-r+1,q+r-2}}$$

where $Z_{-1}^{p,q} := \mathcal{F}^p\Omega^{p+q}(A,V)$. Since $\mathrm{d}_A\, Z_r^{p,q} \subseteq Z_r^{p+r,q-r+1}$, we have induced differentials

$$\mathrm{d}_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The r-th page of the spectral sequence is the total complex

$$\Big(E_r^\bullet = \bigoplus_{p+q=\bullet} E_r^{p,q}, \mathrm{d}_r\Big).$$

The $r+1$-th page is canonically isomorphic to the cohomology of the r-th page:

$$E_{r+1}^{p,q} \simeq \mathrm{H}^{p,q}(E_r^\bullet, \mathrm{d}_r).$$

A filtration as above can be constructed from a **differential graded ideal**

$$\mathcal{I}^\bullet \subseteq \Omega^\bullet(A),$$

by using the powers of $\mathcal{I}$, as follows

$$\mathcal{F}^p_{\mathcal{I}}\Omega^\bullet(A,V) := \big(\wedge^p \mathcal{I} \wedge \Omega(A,V)\big)^\bullet \subseteq \Omega^\bullet(A,V).$$

A Lie subalgebroid $i\colon L \hookrightarrow A$ yields a differential graded ideal

$$\mathcal{I}_L^\bullet := \big(\ker i^*\colon \Omega^\bullet(A) \to \Omega^\bullet(L)\big) \subseteq \Omega^\bullet(A),$$

and so, a filtration

$$\mathcal{F}^p_L\Omega^\bullet(A,V) := \mathcal{F}^p_{\mathcal{I}_L}\Omega^\bullet(A,V) \subseteq \Omega^\bullet(A,V).$$

The corresponding spectral sequence will be called **Serre spectral sequence** associated to the Lie subalgebroid $i\colon L \hookrightarrow A$. If $N \subseteq M$ is the base of L, note that in form degree zero $\mathcal{I}_L^0 = \mathcal{I}_N$ is given by the vanishing ideal of N.

Remark 2.2.1 Likewise, one can use an arbitrary Lie algebroid morphism $\Phi\colon L \to A$ to obtain a differential graded ideal and so a filtration

$$\mathcal{I}_\Phi^\bullet := (\ker \Phi^*)^\bullet \subseteq \Omega^\bullet(A), \qquad \mathcal{F}^p_{\mathcal{I}_\Phi}\Omega^\bullet(A,V) \subseteq \Omega^\bullet(A,V),$$

giving rise to a spectral sequence. While Theorem 2.2.2 can still be formulated and proven in this more general setting, in this chapter we exclusively discuss the Serre spectral sequence arising from Lie subalgebroids.

We have the following result regarding convergence.

Theorem 2.2.2 *Let $i\colon L \hookrightarrow A$ be a Lie subalgebroid of $A \Rightarrow M$ with base $N \subseteq M$ and $V \neq 0_M$ a representation of A. The following are equivalent:*

1. *N is dense in M;*
2. *$\mathcal{I}_L^0 = 0$;*
3. *The filtration $\big\{\mathcal{F}^p_L\Omega^\bullet(A,V)\big\}_{p\geq 0}$ is finite;*
4. *The filtration $\big\{\mathcal{F}^p_L\Omega^\bullet(A,V)\big\}_{p\geq 0}$ is Hausdorff.*

If either condition holds, then

$$\mathcal{F}^p_L\Omega^\bullet(A,V) = 0, \quad \textit{for} \quad p > \bullet.$$

In particular, the Serre spectral sequence stabilises at the page $r+1$, where $r = \mathrm{rank}(A)$, and therefore it converges

$$\mathrm{H}^\bullet(A,V) \simeq \bigoplus_{p+q=\bullet} E_{r+1}^{p,q}.$$

Proof. Condition (a) is equivalent to the restriction $\mathscr{C}^\infty(M) \to \mathscr{C}^\infty(N)$ being injective, which is equivalent to (b).

If (b) holds, then

$$\mathcal{F}^p_L\Omega(A,V) = \wedge^p\mathcal{I}_L \wedge \Omega(A,V) \subseteq \bigoplus_{k=p}^{r} \Omega^k(A,V), \qquad \text{hence} \qquad \mathcal{F}^{r+1}_L\Omega(A,V) = 0.$$

So we obtain (c) and that the spectral sequence stabilises at the page $r+1$.

Clearly, (c) implies (d).

Assume now that N is not dense in M. By the standard construction of bump functions, we find a non-zero function $f \in \mathscr{C}^\infty(M)$, with support in $M \setminus N$, such that $f \geq 0$, and $f^{\frac{1}{p}} \in \mathscr{C}^\infty(M)$, for all $p \geq 1$. Hence $f \in (\mathcal{I}^0_L)^p$, for all $p \geq 0$. Choose $\eta \in \Gamma(V)$ such that $f\eta \neq 0$. It follows that

$$f\eta \in \bigcap_{p\geq 0} \mathcal{F}^p_L\Omega^0(A,V),$$

thus the filtration is not Hausdorff. This shows that (d) implies (a). □

Even if $N \subseteq M$ is not dense, one can make use of the spectral sequence. If $N \subseteq M$ is a closed and embedded submanifold of positive codimension, by Theorem 2.2.2 the induced filtration on $\Omega^\bullet(A)$ is neither Hausdorff nor finite. Instead, the spectral sequence converges to formal cohomology along N of forms on A with values in V.

Theorem 2.2.3 *Let $L \Rightarrow N$ be a Lie subalgebroid of $A \Rightarrow M$ over a closed, embedded submanifold $N \subseteq M$ and fix a representation $V \to M$ of A. The Serre spectral sequence associated to $L \Rightarrow N$ converges to the formal cohomology of A along N with values in V.*

To define formal cohomology, we first recall the notion of jets of sections of a vector bundle. Let a vector bundle $E \to M$ be given and $N \subseteq M$ a closed, embedded submanifold with vanishing ideal $\mathcal{I}_N$. We denote the space of ∞-jets of sections of E along N by

$$\mathscr{J}^\infty_N\Gamma(E) := \Gamma(E)/\mathcal{I}^\infty_N\Gamma(E),$$

where we define $\mathcal{I}^\infty_N := \bigcap_{\ell\geq 0}\mathcal{I}^\ell_N$.

For any closed and embedded submanifold $N \subseteq M$, the ∞-jets of forms on A along N inherit a differential.

Lemma 2.2.4 *The set $\mathcal{I}^\infty_N\Omega^\bullet(A,V) \subseteq \Omega^\bullet(A,V)$ is a differential ideal.*

Proof. By [Nag73, Theorem 1] every function in $\mathcal{I}^\infty_N$ is the product of two functions in $\mathcal{I}^\infty_N$. Then the Leibniz rule of the differential on $\Omega^\bullet(A,V)$ implies the statement. □

Lemma 2.2.4 allows to define **formal cohomology** of a Lie algebroid along a submanifold as the cohomology of the quotient complex $\mathscr{J}^\infty_N\Omega^\bullet(A,V)$. Having this notion clarified, we move on to the proof of Theorem 2.2.3.

Proof of Theorem 2.2.3. For any Lie subalgebroid $L \Rightarrow N$ of $A \Rightarrow M$, the ideal $\mathcal{I}_L$ is given by

$$\mathcal{I}_L \cap \Omega^q(A) = \{\omega \in \Omega^q(A) : i^*\omega = 0\} \supset \mathcal{I}_N \Omega^q(A),$$

which is an equality in degree $q = 0$. Moreover, by counting degrees we find that elements in $\mathcal{F}^p_L \Omega^q(A, V)$ can be written as sums of elements of the form

$$\omega_{i_1} \wedge \ldots \wedge \omega_{i_p} \wedge \eta,$$

where $\omega_{i_j} \in \mathcal{I}_L \cap \Omega^{i_j}(A)$, $q \geq i_1 + \ldots + i_p = k$, $i_1, \ldots, i_p \geq 0$, and $\eta \in \Omega^{q-k}(A, V)$. In particular, if $p > q$, at least $p - q$ of the indices $i_1, \ldots, i_k$ have to be zero, thus $\mathcal{F}^p_L \Omega^q(A, V) \subseteq \mathcal{I}^{p-q}_N \Omega^q(A, V)$. Together, for $p > q$ we obtain

$$\mathcal{I}^p_N \Omega^q(A, V) \subseteq \mathcal{F}^p_L \Omega^q(A, V) \subseteq \mathcal{I}^{p-q}_N \Omega^q(A, V),$$

which implies

$$\bigcap_{p=0}^{\infty} \mathcal{F}^p_L \Omega^\bullet(A, V) = \mathcal{I}^\infty_N \Omega^\bullet(A, V).$$

The rest of the proof is a general argument which applies to spectral sequences corresponding to a filtered complex (see e.g. [Wei94, Exercise 5.4.2]). First, the induced filtration on the quotient complex $\mathscr{J}^\infty_N \Omega^\bullet(A, V)$

$$\hat{\mathcal{F}}^p_L \mathscr{J}^\infty_N \Omega^\bullet(A, V) := \mathcal{F}^p_L \Omega^\bullet(A, V) / \mathcal{I}^\infty_N \Omega^\bullet(A, V) \subseteq \mathscr{J}^\infty_N \Omega^\bullet(A, V)$$

is Hausdorff, and the induced spectral sequence converges to formal cohomology along N of forms of A with values in V. The quotient map $\Omega^\bullet(A, V) \to \mathscr{J}^\infty_N \Omega^\bullet(A, V)$ preserves the respective filtrations and thus induces a map between spectral sequences. This map is an isomorphism on the zeroth page as

$$\hat{E}^{p,q}_0 = \frac{\hat{\mathcal{F}}^p_L \mathscr{J}^\infty_N \Omega^{p+q}(A, V)}{\hat{\mathcal{F}}^{p+1}_L \mathscr{J}^\infty_N \Omega^{p+q}(A, V)} \simeq \frac{\mathcal{F}^p_L \Omega^{p+q}(A, V)}{\mathcal{F}^{p+1}_L \Omega^{p+q}(A, V)} = E^{p,q}_0.$$

By the Mapping Lemma ([Wei94, Lemma 5.2.4], see also [McC01, Theorem 3.5]) the two spectral sequences are isomorphic, showing that the Serre spectral sequence converges to the formal Lie algebroid cohomology along N with values in V. $\square$

In the rest of the section, we discuss in detail the structure on the zeroth page of the Serre spectral sequences in case of a wide Lie subalgebroid (Section 2.2.2, applying Theorem 2.2.2) and a Lie subalgebroid over a closed embedded submanifold (Section 2.2.3, applying Theorem 2.2.3).

2.2.2 Wide Lie subalgebroids

Throughout this section, we fix a Lie algebroid $A \Rightarrow M$, a wide Lie subalgebroid $L \subseteq A$ (i.e. a Lie subalgebroid over the same base), and a representation V of A. By Theorem 2.2.2, the filtration corresponding to L is finite. The filtration can be given a more classical description.

Lemma 2.2.5 *If $p > n$, then $\mathcal{F}^p_L\Omega^n(A,V) = 0$, and if $p \leq n$, then*

$$\mathcal{F}^p_L\Omega^n(A,V) = \{\omega \in \Omega^n(A,V) : \omega(\alpha_1,\ldots,\alpha_n) = 0, \\ \text{if } \alpha_1,\ldots,\alpha_{n-p+1} \in \Gamma(L)\}.$$

Proof. For clarity, choose a vector subbundle $C \subseteq A$ which is a complement of L in A

$$A = L \oplus C.$$

This gives a dual decomposition $A^* = C^{\mathrm{ann}} \oplus L^{\mathrm{ann}}$, where $\cdot^{\mathrm{ann}}$ denotes the annihilator. This induces a decomposition on the level of forms:

$$\Omega^n(A,V) = \bigoplus_{k=0}^{n} \Gamma(\wedge^{n-k}C^{\mathrm{ann}} \otimes \wedge^k L^{\mathrm{ann}} \otimes V).$$

Using the similar decomposition for $\Omega^n(A)$, one obtains that

$$\mathcal{I}^n_L = \bigoplus_{k=1}^{n} \Gamma(\wedge^{n-k}C^{\mathrm{ann}} \otimes \wedge^k L^{\mathrm{ann}}).$$

We have that

$$\wedge^p\mathcal{I}^n_L = \bigoplus_{k=p}^{n} \Gamma(\wedge^{n-k}C^{\mathrm{ann}} \otimes \wedge^k L^{\mathrm{ann}}).$$

That the left-hand side is included in the right-hand side is obvious, the other inclusion follows by applying repeatedly Lemma 2.5.1 (1). A similar argument shows that

$$\mathcal{F}^p_L\Omega^n(A,V) = \bigoplus_{k=p}^{n} \Gamma(\wedge^{n-k}C^{\mathrm{ann}} \otimes \wedge^k L^{\mathrm{ann}} \otimes V). \tag{2.3}$$

This is equivalent to the intrinsic description in the statement. □

We go on to identify the first page of the spectral sequence. For this, recall that L has a canonical representation on the "normal bundle" A/L, induced by the Lie bracket,

$$\nabla\colon \Gamma(L) \times \Gamma(A/L) \to \Gamma(A/L), \quad \nabla_\beta(\overline{\alpha}) := \overline{[\beta,\alpha]}, \tag{2.4}$$

for all $\alpha \in \Gamma(A)$ and $\beta \in \Gamma(L)$. Here $\overline{\delta} \in \Gamma(A/L)$ denotes the image of $\delta \in \Gamma(A)$ under the projection $A \to A/L$. This representation is also called the **Bott-connection**. This induces the dual representation on $(A/L)^*$, the exterior power representation on $\wedge^p(A/L)^*$, and finally, the tensor product representation of L on $\wedge^p(A/L)^* \otimes V$,

$$\nabla\colon \Gamma(L) \times \Gamma(\wedge^p(A/L)^* \otimes V) \to \Gamma(\wedge^p(A/L)^* \otimes V). \tag{2.5}$$

We have the following.

Theorem 2.2.6 *For the first page of the Serre spectral sequence associated to* L*, there is a canonical isomorphism*

$$E_1^{p,q} \simeq \mathrm{H}^q(L, \wedge^p(A/L)^* \otimes V).$$

More precisely, there exists a canonical isomorphism

$$E_0^{p,q} \simeq \Omega^q(L, \wedge^p(A/L)^* \otimes V),$$

under which the differential d_0 *corresponds to the differential of the representation* (2.5)*.*

Proof. The identification follows by using the short exact sequence:

$$0 \to \mathcal{F}_L^{p+1}\Omega^{p+q}(A, V) \to \mathcal{F}_L^p\Omega^{p+q}(A, V) \overset{\mathrm{pr}}{\to} \Omega^q(L, \wedge^p(A/L)^* \otimes V) \to 0,$$

where the map pr acts as

$$\mathrm{pr}(\omega)(\alpha_1, \dots, \alpha_q) := \omega(\alpha_1, \dots, \alpha_q, \,\cdot\,, \dots, \,\cdot\,) \in \Gamma(\wedge^p(A/L)^* \otimes V),$$

and we use the canonical isomorphism $(A/L)^* \simeq L^{\mathrm{ann}}$. That the above is indeed a short exact sequence follows immediately from the definitions. A direct calculation (see e.g. the proof of [Mac05, Proposition 7.4.3]) implies the statement about the differentials. ☐

The description of the filtration from Lemma 2.2.5 shows that the Serre spectral sequence generalises classical constructions.

Example 2.2.7 Let $\mathfrak{g}$ be a Lie algebra and V a representation of $\mathfrak{g}$. The filtration induced by a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is given by

$$\mathcal{F}_{\mathfrak{h}}^p \wedge^{p+q} \mathfrak{g}^* \otimes V = (\wedge^p \mathfrak{h}^{\mathrm{ann}}) \wedge (\wedge^q \mathfrak{g}^*) \otimes V,$$

and the resulting spectral sequence coincides with the Hochschild-Serre spectral sequence for Lie algebras [HS53]. By Theorem 2.2.6, the first page is given by

$$E_1^{p,q} \simeq \mathrm{H}^q(\mathfrak{h}, \wedge^p(\mathfrak{g}/\mathfrak{h})^* \otimes V).$$

Example 2.2.8 A representation of $TM \Rightarrow M$ is the same as a bundle $V \to M$ endowed with a flat connection. The Lie algebroid cohomology of TM with coefficients in V can be understood as de Rham cohomology of M with **local/twisted coefficients**. A surjective submersion $\pi\colon M \to Q$ yields a wide subalgebroid $\ker T\pi \subseteq TM$. The induced filtration on $\Omega^\bullet(M, V)$ is given by

$$\begin{aligned}\mathcal{F}_\pi^p\Omega^{p+q}(M, V) = \{\omega \in \Omega^{p+q}(M, V) :& \omega(v_1, \dots, v_{p+q}) = 0, \\ & \text{if } v_1, \dots, v_{q+1} \in \ker T\pi\}.\end{aligned}$$

The resulting spectral sequence coincides with the Leray–Serre spectral sequence in de Rham cohomology with local coefficients in V, which was carefully developed in [Hat60]. We will come back to this example in Subsection 2.3.1.

Example 2.2.9 By Frobenius' Theorem, wide Lie subalgebroids of TM are the same as foliations $\mathcal{F}$ on M. Denote the tangent bundle of a foliation $\mathcal{F}$ by $T\mathcal{F} \Rightarrow M$. Forms on the Lie algebroid $T\mathcal{F}$ are called **foliated forms** and will be denoted by $\Omega^\bullet(\mathcal{F})$. The representation (2.4) becomes the Bott connection of $T\mathcal{F}$ on the normal bundle $\nu_{\mathcal{F}} = TM/T\mathcal{F}$.

The Serre spectral sequence of the inclusion $T\mathcal{F} \subseteq TM$ converges to the cohomology of M, and Theorem 2.2.6 shows that its first page is given by

$$E_1^{p,q} \simeq \mathrm{H}^q(\mathcal{F}, \wedge^p \nu_{\mathcal{F}}^*).$$

For later use, we describe the differential on the first page in position on $E_1^{0,q}$, i.e.

$$\mathrm{d}_1 : \mathrm{H}^q(\mathcal{F}) \to \mathrm{H}^q(\mathcal{F}, \nu_{\mathcal{F}}^*).$$

Let $c \in \mathrm{H}^q(\mathcal{F})$, with representative a foliated q-form $\eta \in \Omega^q(\mathcal{F})$, i.e. η is a smoothly varying family of closed q-forms on the leaves of $\mathcal{F}$. Let $\tilde{\eta} \in \Omega^q(M)$ be an extension of η to a q-form on M. Since the pullback of $\mathrm{d}\tilde{\eta}$ to the leaves of $\mathcal{F}$ vanishes, we obtain an element

$$(\mathrm{d}\tilde{\eta})_{1,q} \in \Omega^q(\mathcal{F}, \nu_{\mathcal{F}}^*), \quad (\mathrm{d}\tilde{\eta})_{1,q}(X_1, \dots, X_q) := (\mathrm{d}\tilde{\eta})(X_1, \dots, X_q, \,\cdot\,) \in \nu_{\mathcal{F}}^*,$$

where we regard $\nu_{\mathcal{F}}^*$ as the annihilator of $T\mathcal{F}$. Moreover, $(\mathrm{d}\tilde{\eta})_{1,q}$ is closed for the complex computing foliated cohomology with values in $\nu_{\mathcal{F}}^*$, and its class is independent of the extension $\tilde{\eta}$ of η or even on the chosen representative η of c. With these, we have that

$$\mathrm{d}_1[\eta] = [(\mathrm{d}\tilde{\eta})_{1,q}] \in \mathrm{H}^q(\mathcal{F}, \nu_{\mathcal{F}}^*).$$

In the generality of Theorem 2.2.6, not much more can be said about the Serre spectral sequence of wide Lie subalgebroids. Starting from Section 2.3, we will restrict to the class of Lie subalgebroids that are kernels of surjective morphisms, which will enable us to reveal more information about their associated Serre spectral sequence.

2.2.3 Lie subalgebroids over closed submanifolds

In this section, we fix a Lie algebroid $A \Rightarrow M$, a Lie subalgebroid $L \subseteq A$ over a closed, embedded submanifold $N \subseteq M$, and a representation V of A. By Theorem 2.2.3, we obtain a spectral sequence converging to the formal cohomology of A along N with values in V, for which we have the following.

Theorem 2.2.10 *There exists an isomorphism*

$$E_1^{p,q} \simeq \mathrm{H}(L, \wedge^p(\nu_N^* \oplus L^{\mathrm{ann}}) \otimes V\big|_N)^q,$$

where $\nu_N^* \oplus L^{\mathrm{ann}}$ is the representation up to homotopy of L corresponding to the VB-algebroid

$$\begin{tikzcd}
\nu_L(A)^* \arrow[r, Rightarrow] \arrow[d] & L^{\mathrm{ann}} \arrow[d] \\
L \arrow[r, Rightarrow] & N
\end{tikzcd}$$

Remark 2.2.11 We denote the normal bundle of a closed, embedded submanifold $Y \subseteq X$ by

$$\nu_Y(X) = TX\big|_Y / TY,$$

or ν_Y if the ambient manifold X is clear from context.

Remark 2.2.12 As we will discuss below, $\nu_L(A)^* \to L$ is canonically a VB-algebroid, and therefore, with the aid of a splitting, it can be regarded as a two-term representation up to homotopy of L [GSM10, Theorem 4.11].

Remark 2.2.13 Theorem 2.2.10 can be seen as a generalisation of Theorem 2.2.6, since for $N = M$ the representation up to homotopy of L with values in $\wedge^p L^{\mathrm{ann}} \otimes V$ coincides with the classical representation (2.4).

After recalling some of the necessary tools and objects, we will prove Theorem 2.2.10.

First, we recall the notion of a representation up to homotopy [AC13, Definition 3.1].

Definition 2.2.14 A **representation up to homotopy** of $L \Rightarrow N$ on a $\mathbb{Z}$-graded vector bundle $E^\bullet$ over the same base N is a differential

$$\mathrm{D}\colon \Omega(L,E)^\bullet \to \Omega(L,E)^{\bullet+1},$$

where the total degree is defined as

$$\Omega(L,E)^q = \bigoplus_{i+j=q} \Omega^i(L,E^j),$$

satisfying, for any $\omega \in \Omega^k(L)$ and $\eta \in \Omega(L,E)^\bullet$, the Leibniz rule

$$\mathrm{D}(\omega \wedge \eta) = \mathrm{d}_L\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{D}\eta. \tag{2.6}$$

Note that the differential of a representation up to homotopy of L on $E^\bullet$ is determined by its values on $\Omega^0(L,E^\bullet) = \Gamma(E^\bullet)$, because of the Leibniz rule (2.6). Let us also note that one can construct duals and tensor powers of representations up to homotopy [AC13, Section 4].

Next we quickly recall the notion of a VB-algebroid and the construction of the corresponding representation up to homotopy following [GSM10]. Let a double vector bundle

$$\begin{tikzcd} B \arrow[r] \arrow[d] & E^0 \arrow[d] \\ L \arrow[r] & N \end{tikzcd} \tag{2.7}$$

with core $E^{-1} := \ker \pi_{B\to E^0} \cap \ker \pi_{B\to L}$ be given. The bundle $B \to E^0$ has two distinguished classes of sections, called **linear** and **core sections**,

$$\Gamma_{\mathrm{core}}(B) := \{c \circ \pi_{E^0\to N} +_L 0_{E^0\to B} : c \in \Gamma(E^{-1})\} \subseteq \Gamma(B \to E^0)$$

$$\Gamma_{\mathrm{lin}}(B) := \{b : b\colon E^0 \to B \text{ is a vector bundle morphism}\} \subseteq \Gamma(B \to E^0).$$

Two Lie algebroid structures $B \Rightarrow E^0$ and $L \Rightarrow N$ yield a **VB-algebroid** structure on the double vector bundle B if and only if the following compatibility conditions hold:

$$\begin{aligned} [\Gamma_{\text{lin}}(B), \Gamma_{\text{lin}}(B)] &\subseteq \Gamma_{\text{lin}}(B), \\ [\Gamma_{\text{lin}}(B), \Gamma_{\text{core}}(B)] &\subseteq \Gamma_{\text{core}}(B), \text{ and} \\ [\Gamma_{\text{core}}(B), \Gamma_{\text{core}}(B)] &= 0. \end{aligned}$$

To describe how the VB-algebroid (2.7) encodes a representation up to homotopy of L on $E^\bullet = E^{-1} \oplus E^0$, we recall that linear sections are sections of a Lie algebroid denoted by $\hat{L} \to N$. This Lie algebroid fits into a short exact sequence

$$0 \to \operatorname{Hom}(E^0, E^{-1}) \to \hat{L} \to L \to 0. \tag{2.8}$$

In (2.8), the map $\hat{L} \to L$ is given on sections by projecting a linear section to its base map, which is necessarily a section of L. Choosing a splitting $\sigma\colon L \to \hat{L}$ of (2.8) allows to define

- An L-connection $\nabla^{E^{-1}}$ on E^{-1} given by

  $$\nabla_a^{E^{-1}}(c) = [\sigma(a), c]_B$$

  for $a \in \Gamma(L)$ and $c \in \Gamma(E^{-1}) = \Gamma_{\text{core}}(B)$.

- An L-connection ∇^{E^0} on E^0 with dual connection given by

  $$\nabla_a^{(E^0)^*}\xi = \sharp_B(\sigma(a))\xi,$$

  where $a \in \Gamma(L)$ and $\xi \in \Gamma((E^0)^*)$ is considered a linear function on E^0.

- A $\operatorname{Hom}(E^0, E^{-1})$-valued two-form γ on L given by the curvature of σ, i.e.

  $$\gamma(a_1, a_2) := [\sigma(a_1), \sigma(a_2)] - \sigma([a_1, a_2])$$

  for $a_1, a_2 \in \Gamma(L)$.

- Finally, independent of the splitting, there is the core map $\partial : E^{-1} \to E^0$, defined to be minus the anchor of B from the core of B to the core E^0 of TE^0.

These maps piece together to the restriction of the differential D of $\Omega(L, E)^\bullet$ to $\Gamma(E^\bullet)$.

Example 2.2.15 For a Lie subalgebroid $L \Rightarrow N$ of $A \Rightarrow M$ there is a VB algebroid

$$\begin{tikzcd} \nu_L(A) \arrow[r, Rightarrow] \arrow[d] & \nu_N(M) \arrow[d] \\ L \arrow[r, Rightarrow] & N \end{tikzcd} \tag{2.9}$$

with $E^0 = \nu_N(M)$ and core $E^{-1} = A\big|_N/L$ [MP21]. The bracket of $\nu_L(A) \Rightarrow \nu_N(M)$ is defined such that the map induced by the normal bundle functor

$$\begin{aligned} \nu\colon \Gamma(A,L) &\to \Gamma(\nu_L(A)),\\ \nu(a)(X \operatorname{mod} TN) &= (\mathrm{d}a)(X) \operatorname{mod} TL, \quad X \in TM\big|_N \end{aligned} \tag{2.10}$$

is bracket preserving, where $\Gamma(A,L)$ denotes the set of sections of A that restrict to L along N. The image of the map ν is precisely the set of linear sections. To obtain a splitting and identify the core sections, we choose a vector bundle isomorphism $A\big|_E \simeq \mathrm{pr}^* A\big|_N$, where $\mathrm{pr}\colon E \to N$ is a tubular neighbourhood, and we choose a complement $A\big|_N = L \oplus C$. Then core sections are sections of $C \simeq E^{-1}$. We define a splitting of (2.8) by using the map ν from (2.10), via

$$\sigma\colon \Gamma(L) \to \Gamma(\hat{L}), \qquad a \mapsto \nu(\mathrm{pr}^* a).$$

The L-connections are given as follows. For $a \in \Gamma(L)$ and $c \in \Gamma(C)$,

$$\nabla^C_a c = \mathrm{pr}_C[\mathrm{pr}^* a, \mathrm{pr}^* c]\big|_N \tag{2.11}$$

and for $a \in \Gamma(L)$ and $f \in \mathcal{I}_N$, with $\mathrm{d}f\big|_N \in \Gamma(\nu_N^*)$,

$$\nabla^{\nu_N^*}_a \mathrm{d}f\big|_N = \mathrm{d}(\sharp(\mathrm{pr}^* a) f)\big|_N. \tag{2.12}$$

Remark 2.2.16 Given a VB-algebroid as in (2.7) one can dualise over L to obtain a new VB-algebroid

$$\begin{tikzcd} B^* \arrow[r, Rightarrow] \arrow[d] & (E^{-1})^* \arrow[d] \\ L \arrow[r, Rightarrow] & N \end{tikzcd}$$

with core $(E^0)^*$. If σ is a splitting of (2.7) with curvature γ, then the structure maps corresponding to the representation up to homotopy induced by the dual VB-algebroid are given by $-\partial^*$, $\nabla^{(E^0)^*}$, $\nabla^{(E^{-1})^*}$ and $-\gamma^*$. Note that the different signs compared to [AC13, Example 4.1] come from a degree shift.

Example 2.2.17 Taking the dual of the VB-algebroid $\nu_L(A) \to L$, we obtain the conormal VB-algebroid

$$\begin{tikzcd} \nu_L(A)^* \arrow[r, Rightarrow] \arrow[d] & L^{\mathrm{ann}} \arrow[d] \\ L \arrow[r, Rightarrow] & N \end{tikzcd}$$

with core ν_N^*. Fix a tubular neighbourhood $\mathrm{pr}\colon E \to N$ of N in M, a vector bundle isomorphism $A\big|_E \simeq \mathrm{pr}^*(A\big|_N)$ and a complement $A\big|_N = L \oplus C$. Then we obtain a representation up to homotopy of L on the graded bundle

$$\nu_N^* \oplus L^{\mathrm{ann}},$$

where $\deg(\nu_N^*) = -1$ and $\deg(L^{\mathrm{ann}}) = 0$, and the differential is defined as explained in Example 2.2.15 and Remark 2.2.16. Taking the (graded) exterior power of this representation up to homotopy, in the sense of [AC13], we obtain a representation up to homotopy of L on the graded vector bundle

$$\wedge^p(\nu_N^* \oplus L^{\mathrm{ann}}) = \bigoplus_{k=0}^{p} S^k(\nu_N^*) \otimes \wedge^{p-k}(L^{\mathrm{ann}}),$$

where $\deg(S^k(\nu_N^*) \otimes \wedge^{p-k}(L^{\mathrm{ann}})) = -k$.

Lemma 2.2.18 *Using the notation and choices from Example 2.2.17, we can identify*

$$E_0^{p,q} \simeq \bigoplus_{i=0}^{p} \Omega^{p+q-i}(L, S^{p-i}(\nu_N^*) \otimes \wedge^i(L^{\mathrm{ann}}) \otimes V\big|_N). \tag{2.13}$$

The differential induced by d_0 *satisfies a Leibniz rule, turning*

$$(E_0^{p,\bullet}, \mathrm{d}_0) \simeq (\Omega(L, \wedge^p(\nu_N^* \oplus L^{\mathrm{ann}}) \otimes V\big|_N)^\bullet, \mathrm{D})$$

into a representation of L *up to homotopy.*

Proof. Over E we decompose $A\big|_E = \mathrm{pr}^*L \oplus \mathrm{pr}^*C$, and then forms on $A\big|_E$ decompose as

$$\Omega^k(A\big|_E, V\big|_E) = \bigoplus_{i=0}^{k} \Gamma(\wedge^i \mathrm{pr}^*C^* \otimes \wedge^{k-i}\mathrm{pr}^*L^* \otimes \mathrm{pr}^*V\big|_N),$$

as in the proof of Lemma 2.2.5. Moreover, note that the ideal $\mathcal{I}_L$, i.e. the kernel of the pullback map $i^*\colon \Omega^\bullet(A\big|_E) \to \Omega^\bullet(L)$, is generated by $\mathcal{I}_N$ and $\Omega^1(\mathrm{pr}^*C)$. From this it follows that

$$\mathcal{F}_L^p\Omega^k(A\big|_E, V\big|_E) = \bigoplus_{i=0}^{k} \mathcal{I}_N^{p-i}\Gamma(\wedge^i \mathrm{pr}^*C^* \otimes \wedge^{k-i}\mathrm{pr}^*L^* \otimes \mathrm{pr}^*V\big|_N).$$

Equation (2.13) follows by using the canonical isomorphism

$$\mathcal{I}_N^p\Gamma(F)/\mathcal{I}_N^{p+1}\Gamma(F) \simeq \Gamma(S^p\nu_N^* \otimes F\big|_N),$$

which hold for any vector bundle $F \to M$, and the identification

$$\frac{\mathcal{F}_L^p\Omega^\bullet(A, V)}{\mathcal{F}_L^{p+1}\Omega^\bullet(A, V)} \simeq \frac{\mathcal{F}_L^p\Omega^\bullet(A\big|_E, V\big|_E)}{\mathcal{F}_L^{p+1}\Omega^\bullet(A\big|_E, V\big|_E)}.$$

Finally, we check the Leibniz rule (2.6). For $\eta \in \mathcal{F}_L^p\Omega^\bullet(A\big|_E, V\big|_E)$ and $\omega \in \Omega^k(L)$, we have

$$\begin{aligned}
\mathrm{d}_0(\omega \wedge [\eta]) &= [\mathrm{d}_A(\mathrm{pr}^*\omega \wedge \eta)] \\
&= [\mathrm{d}_A(\mathrm{pr}^*\omega) \wedge \eta + (-1)^k \mathrm{pr}^*\omega \wedge \mathrm{d}_A\eta] \\
&= [\mathrm{pr}^*\mathrm{d}_L(\omega) \wedge \eta + (-1)^k \mathrm{pr}^*\omega \wedge \mathrm{d}_A\eta + \underbrace{(\mathrm{d}_A\mathrm{pr}^*\omega - \mathrm{pr}^*\mathrm{d}_L\omega)}_{\in \mathcal{I}_L} \wedge \eta] \\
&= \mathrm{d}_L(\omega) \wedge [\eta] + (-1)^k \omega \wedge \mathrm{d}_0[\eta].
\end{aligned}$$

□

To complete the proof of Theorem 2.2.10, we need to show that the differential we obtain via the identification (2.13) is given by graded antisymmetric powers of the representation up to homotopy described in Example 2.2.17.

Proof of Theorem 2.2.10. First we determine the differential on

$$E_0^{1,\bullet} = \Omega^{\bullet+1}(L, \nu_N^* \otimes V\big|_N) \oplus \Omega^\bullet(L, L^{\mathrm{ann}} \otimes V\big|_N)$$

under the identification (2.13). For $f \in \mathcal{I}_N$ and $v \in \Gamma(V\big|_N)$ we obtain

$$\mathrm{d}_0(\mathrm{d}f\big|_N \otimes v) \in \Omega^1(L, \nu_N^* \otimes V\big|_N) \oplus \Gamma(L^{\mathrm{ann}} \otimes V\big|_N)$$

for degree reasons. To compute the first summand, let $a \in \Gamma(L)$ be given. Then

$$\begin{aligned} \mathrm{d}_0(\mathrm{d}f\big|_N \otimes v)(a) &= [\mathrm{d}_A(f)(\mathrm{pr}^*a) \otimes \mathrm{pr}^*v + f\nabla^V_{\mathrm{pr}^*a}\mathrm{pr}^*v] \\ &= \mathrm{d}(\sharp(\mathrm{pr}^*a)f)\big|_N \otimes v + \mathrm{d}f\big|_N \otimes \nabla_a^{V|_N} v, \end{aligned}$$

which is (2.12). To compute the term in $\Gamma(L^{\mathrm{ann}} \otimes V\big|_N)$, let $c \in \Gamma(C)$ be given. Then

$$\begin{aligned} \mathrm{d}_0(\mathrm{d}f\big|_N \otimes v)(c) &= \mathrm{d}_A(f \otimes \mathrm{pr}^*v)\big|_N(c) \\ &= \mathrm{d}f\big|_N(\sharp c) \otimes v + f\big|_N \nabla^V_{\mathrm{pr}^*c}\mathrm{pr}^*v\big|_N \\ &= \sharp^*(\mathrm{d}f\big|_N)(c) \otimes v. \end{aligned}$$

Now let $\gamma \in \Gamma(L^{\mathrm{ann}})$ be given. Then

$$\mathrm{d}_0(\gamma \otimes v) \in \Omega^1(L, L^{\mathrm{ann}}) \oplus \Omega^2(L, \nu_N^*).$$

Thus, let $a, b \in \Gamma(L)$ and $c \in \Gamma(C)$ be given. Then

$$\begin{aligned} \mathrm{d}_0(\gamma \otimes v)(a)(c) =& \mathrm{d}_A(\mathrm{pr}^*\gamma \otimes \mathrm{pr}^*v)(\mathrm{pr}^*a, \mathrm{pr}^*\mathrm{c}) \\ =& \sharp(\mathrm{pr}^*a)(\mathrm{pr}^*\gamma(c)) \otimes \mathrm{pr}^*v\big|_N + \mathrm{pr}^*\gamma(c)\nabla^V_{\mathrm{pr}^*a}\mathrm{pr}^*v\big|_N \\ & - \mathrm{pr}^*\gamma([\mathrm{pr}^*a, \mathrm{pr}^*c])\big|_N \\ =& \sharp a(\gamma(c)) + \gamma(c)\nabla_a^{V|_N} v - \gamma([\mathrm{pr}^*a, \mathrm{pr}^*c]\big|_N)v, \end{aligned}$$

which is dual to (2.11). Finally, for the contribution in $\Omega^2(L, \nu_N^*)$ we find

$$\begin{aligned} \mathrm{d}_0(\gamma \otimes v)(a, b) &= -\mathrm{d}(\mathrm{pr}^*\gamma([\mathrm{pr}^*a, \mathrm{pr}^*b]))\big|_N \otimes v \\ &= -\mathrm{d}\mathrm{pr}^*\gamma([\mathrm{pr}^*a, \mathrm{pr}^*b] - \mathrm{pr}^*[a, b])\big|_N \otimes v, \end{aligned}$$

which shows that $(E_0^{1,\bullet}, \mathrm{d}_0)$ is indeed dual of the representation up to homotopy given by the dual of (2.9) by Remark 2.2.16.

To complete the proof of Theorem 2.2.10, we check that the differential on $E_0^{p,\bullet}$ for $p > 1$ is given by tensor powers of the differential on $E_0^{1,\bullet}$. Again, by the Leibniz rule (2.6) it is enough to calculate d_0 on $\Gamma(\wedge^p(\nu_N^* \oplus L^{\mathrm{ann}}) \otimes V\big|_N)$.

Let $f \in \mathcal{I}_N$ and $\eta \in \mathcal{F}_L^{p-1}\Omega(A,V)^\bullet$ be given. Then $f\eta \in \mathcal{F}_L^p\Omega(A,V)^\bullet$ and, using the Leibniz rule of d_A,

$$\begin{aligned} \mathrm{d}_0[f\eta] &= [\mathrm{d}_A(f\eta)] \\ &= [\mathrm{d}_A f \wedge \eta + f\mathrm{d}_A\eta] \\ &= \mathrm{d}_0[f] \wedge [\eta] + [f] \wedge \mathrm{d}_0[\eta]. \end{aligned}$$

Similarly, a graded Leibniz rule holds for $\gamma\wedge$, where $\gamma \in \Gamma(L^{\mathrm{ann}})$. □

Invariant submanifolds

In this section, let $N \subseteq M$ be a closed, embedded and invariant submanifold of $A \Rightarrow M$, i.e.

$$\sharp(A\big|_N) \subseteq TN.$$

In this case, $L = A\big|_N$ is a subalgebroid, and $L^{\mathrm{ann}} = 0_N$. Moreover, the filtration becomes

$$\mathcal{F}_L^p\Omega^\bullet(A,V) = \mathcal{I}_N^p\Omega^\bullet(A,V). \tag{2.14}$$

The VB-algebroid $\nu_{A|_N}(A) \Rightarrow \nu_N$ (recall Example 2.2.15) is the **action Lie algebroid**

$$A\big|_N \ltimes \nu_N \Rightarrow \nu_N,$$

corresponding to the canonical representation of $A\big|_N$ on the normal bundle ν_N, given by

$$\nabla_a(X\big|_N \bmod TN) = [\sharp\tilde{a}, X]\big|_N \bmod TN$$

for $X \in \Gamma(TM)$ and $\tilde{a} \in \Gamma(A)$ some extension of $a \in \Gamma(A\big|_N)$. We briefly recall the construction of the action Lie algebroid for our case, following [HM90, Theorem 2.4]. Writing $\mathrm{pr}\colon \nu_N \to N$ for the projection, the vector bundle structure is $A\big|_N \ltimes \nu_N := \mathrm{pr}^*A\big|_N \to \nu_N$. On pullback sections the anchor $\sharp_\ltimes : A\big|_N \ltimes \nu_N \to T\nu_N$ is defined by

$$[\sharp_\ltimes(\mathrm{pr}^*a), V^{\mathrm{ver}}] = \big(\nabla_a V\big)^{\mathrm{ver}}$$

for $a \in \Gamma(A\big|_N)$ and $V \in \Gamma(\nu_N)$. Here, we denote by

$$\cdot^{\,\mathrm{ver}} \colon \Gamma(\nu_N) \to \Gamma(T\nu_N)$$

the vertical lift. In particular, $\sharp_\ltimes$ maps pullback sections into linear vector fields, i.e. vector fields of homogeneous degree 0. The bracket $[\cdot\,,\cdot]_\ltimes$ on $\Gamma(A\big|_N \ltimes \nu_N)$ is defined on pullback sections by

$$[\mathrm{pr}^*a, \mathrm{pr}^*b]_\ltimes = \mathrm{pr}^*([a,b]_{A|_N}),$$

where $a, b \in \Gamma(A\big|_N)$, and is extended to all sections using the Leibniz rule.

The action Lie algebroid is linear in the following sense.

Lemma 2.2.19 *For $\lambda \in \mathbb{R}\setminus\{0\}$ the scalar multiplication by λ on ν_N induces a Lie algebroid automorphism*

$$m_\lambda : A\big|_N \ltimes \nu_N \xrightarrow{\sim} A\big|_N \ltimes \nu_N.$$

Proof. First note that for all $a \in \Gamma(A\big|_N)$ we have

$$m_\lambda^* \sharp_\ltimes(\mathrm{pr}^* a) = \sharp_\ltimes(\mathrm{pr}^* a)$$

since $\sharp_\ltimes(\mathrm{pr}^* a)$ is a linear vector field. Then compatibility with the bracket follows immediately since it clearly holds on pullback sections. □

For later use, we introduce the following notion.

Definition 2.2.20 A Lie algebroid $A \Rightarrow M$ is called **linearisable** around the invariant submanifold $N \subseteq M$ if A is isomorphic in a neighbourhood of N to the action Lie algebroid $A\big|_N \ltimes \nu_N \Rightarrow \nu_N$ restricted to a neighbourhood of N.

The dual VB-algebroid $\nu_{A\big|_N}(A)^* \Rightarrow N$ is the semi-direct product associated to the conormal representation of $A\big|_N$ on

$$\nu_N^* \to N.$$

The corresponding representation up to homotopy is therefore the classical representation on ν_N^*, and thus, we obtain the following simplification of Theorem 2.2.10.

Theorem 2.2.21 *Let $N \subseteq M$ be a closed, embedded submanifold which is invariant for $A \Rightarrow M$. The Serre spectral sequence of the Lie subalgebroid $L = A\big|_N$ converges to the formal Lie algebroid cohomology $\mathrm{H}^\bullet(\mathscr{J}_N^\infty \Omega(A, V))$, and its first page is given by*

$$E_1^{p,q} \simeq \mathrm{H}^{p+q}(A\big|_N, S^p \nu_N^* \otimes V\big|_N).$$

This theorem has a straightforward consequence.

Corollary 2.2.22 *If $\mathrm{H}^k(A\big|_N, S^p\nu_N^* \otimes V\big|_N) = 0$ for all $p \geq 0$, then*

$$\mathrm{H}^k(\mathscr{J}_N^\infty \Omega(A, V)) = 0.$$

The assumptions of the corollary hold, for example in the following cases:

- N is a point, the Lie algebra $\mathfrak{g} = A\big|_N$ is semisimple and $k = 1$ or $k = 2$. This is the Whitehead Lemma.
- A is the Lie algebroid of a Hausdorff Lie groupoid $\mathcal{G} \rightrightarrows N$, such that: (1) $\mathcal{G}$ is proper, i.e. the map $(t, s) : \mathcal{G} \to M \times M$ is proper; (2) the fibres of t have zero de Rham cohomology in degree i, with $1 \leq i \leq k$; and (3) $V\big|_N$ integrates to a representation of $\mathcal{G}$. See [Cra00, Proposition 1 & Theorem 4].
- A is the Lie algebroid of a Hausdorff Lie groupoid $\mathcal{G} \rightrightarrows N$, such that: (1) the target map $t : \mathcal{G} \to N$ is proper; (2) the fibres of t have zero de Rham cohomology in degree k; and (3) $V\big|_N$ integrates to a representation of $\mathcal{G}$. See [Mat14, Lemma C.1].

Remark 2.2.23 Formal cohomology plays an important role in formal linearisation problems in Poisson geometry (see Section 2.3.3 for more on Poisson geometry, including definitions and references). A Poisson manifold (M, w) has an associated cotangent Lie algebroid $A = T^*M$, whose cohomology is called the Poisson cohomology of (M, w). We say that w is **linearisable** around an invariant submanifold $N \subseteq M$ (also called a Poisson submanifold), if the Lie algebroid T^*M is linearisable around N (for leaves, this is equivalent to the more standard notion of linearisation discussed in Section 2.3.3, see [FM23, Theorem 8.2]). In degree two, Poisson cohomology encodes infinitesimal deformations of the Poisson structure. Therefore, the formal infinitesimal rigidity of w around the Poisson submanifold $N \subseteq M$ translates to

$$\mathrm{H}^2(\mathscr{J}_N^\infty \Omega(T^*M)) = 0. \tag{2.15}$$

Formal rigidity was studied first around points, i.e. when $N = \{x\}$, and it was shown that if $\mathfrak{g} := A_x$ is semisimple, then the Poisson structure w is formally linearisable around x [Wei83, Theorem 6.1]. This result was extended to symplectic leaves in [IKV98, Theorem 7.1], and to general Poisson submanifolds N in [Mat12, Theorem 1.1], where it was shown that Poisson structures satisfying $\mathrm{H}^2(T^*M\big|_N, S^p\nu_N^*) = 0$ for all p are formally rigid. Corollary 2.2.22 can be seen as the infinitesimal version of this result, as it gives the infinitesimal version of formal rigidity, i.e. that (2.15) holds.

Next we show that if the Lie algebroid is linearisable around the invariant submanifold, then the Serre spectral sequence stabilises at E_1.

Theorem 2.2.24 *Let $N \subseteq M$ be a closed, embedded submanifold which is invariant for $A \Rightarrow M$. Suppose that A is linearisable around N. Then the Serre spectral sequence associated to the subalgebroid $L = A\big|_N$ stabilises at E_1. Hence, the formal algebroid cohomology is given by*

$$\mathrm{H}^\bullet(\mathscr{J}_N^\infty \Omega^\bullet(A)) \simeq \prod_{j=0}^{\infty} \mathrm{H}^\bullet(A\big|_N, S^j\nu_N^*).$$

Proof. The formal cohomology of A along N is canonically isomorphic to the formal cohomology of $A\big|_U$, for any neighbourhood U of N. Since $A \Rightarrow M$ is linearisable around N, we may therefore assume that $A = A\big|_N \ltimes \nu_N$. To deduce the result, it suffices to show that the short exact sequence of cochain complexes

$$0 \to \mathcal{F}^{p+1}_{A|_N} \Omega^{p+q}(A) \to \mathcal{F}^p_{A|_N} \Omega^{p+q}(A) \to E_0^{p,q} \to 0 \tag{2.16}$$

admits a splitting $\sigma : E_0^{p,q} \to \mathcal{F}^p_{A|_N} \Omega^{p+q}(A)$ which is compatible with the differentials. Indeed, if that is the case, then, if $\eta \in E_0^{p,q}$ is d_0-closed, then $\sigma(\eta)$ is d_A-closed. Thus, all subsequent differentials d_r, $r > 0$ are zero, $E_\infty^{p,q} = E_1^{p,q}$ and the statement follows.

To build the splitting, we use the canonical identification

$$\mathrm{Pol}\colon \Gamma(S^p\nu_N^*) \xrightarrow{\sim} \mathrm{Pol}^p(\nu_N),$$

where $\mathrm{Pol}^p(\nu_N) \subseteq \mathscr{C}^\infty(\nu_N)$ denotes the homogeneous polynomials of degree p on ν_N. This yields the splitting $\sigma := \mathrm{pr}^* \otimes \mathrm{Pol}\colon \Omega^{p+q}(A\big|_N, S^p\nu_N^*) \to \Omega^{p+q}(A)$ of (2.16). To show that σ is compatible with the differentials, it suffices to show that its image is a subcomplex. For this, note that the image of σ can be characterised as forms $\theta \in \Omega^{p+q}(A)$ such that,

$$m_\lambda^*\theta = \lambda^p\theta, \qquad \text{for all} \qquad \lambda \in \mathbb{R}\backslash\{0\}.$$

Since m_λ is an automorphism of A (Lemma 2.2.19), it follows that $\mathrm{d}_A \circ m_\lambda^* = m_\lambda^* \circ \mathrm{d}_A$. Therefore, the image of σ is a subcomplex. ☐

Finite jets along invariant submanifolds

In this subsection we discuss a version of the spectral sequence for finite order jets.

Let $N \subseteq M$ be a closed, embedded invariant submanifold of $A \Rightarrow M$. For $k \in \mathbb{N}_0$, the space of k-th order jets along N of forms on A with values in V is defined as

$$\mathscr{J}_N^k\Omega^\bullet(A, V) = \Omega^\bullet(A, V)/\mathcal{I}_N^{k+1}\Omega^\bullet(A, V).$$

By (2.14), $\mathcal{I}_N^{k+1}\Omega^\bullet(A, V)$ is a differential graded $\Omega^\bullet(A)$-submodule of $\Omega^\bullet(A, V)$. Consequently, there is an induced differential on $\mathscr{J}_N^k\Omega^\bullet(A, V)$, giving rise to cohomology of finite jets.

If $L \Rightarrow N$ is a Lie subalgebroid, the inclusion induces a filtration on $\mathscr{J}_N^k\Omega^\bullet(A, V)$ by

$$\mathcal{F}_L^p\mathscr{J}_N^k\Omega^\bullet(A, V) := \mathcal{F}_L^p\Omega^\bullet(A, V)/(\mathcal{I}_N^{k+1}\Omega^\bullet(A, V) \cap \mathcal{F}_L^p\Omega^\bullet(A, V)). \tag{2.17}$$

The corresponding spectral sequence always converges.

Theorem 2.2.25 *Let $L \Rightarrow N$ be a Lie subalgebroid of $A \Rightarrow M$ over the closed, embedded, and invariant submanifold $N \subseteq M$. Fix $k \in \mathbb{N}_0$. The spectral sequence induced by the filtration* (2.17) *stabilises at page $r + 1$, where $r := k + \mathrm{rank}(A) - \mathrm{rank}(L)$, and converges to $\mathrm{H}^\bullet(\mathscr{J}_N^k\Omega(A, V))$. The zeroth page is given by*

$$E_0^{p,q} \simeq \bigoplus_{i=\max\{0,p-k\}}^{p} \Omega^{p+q-i}(L, S^{p-i}\nu_N^* \otimes \wedge^i L^{\mathrm{ann}} \otimes V\big|_N)$$

with d_0 corresponding to the representation up to homotopy of L on $\wedge^p(\nu_N^* \oplus L^{\mathrm{ann}}) \otimes V\big|_N$.

Proof. We restrict to a tubular neighbourhood $E \subseteq M$ of N. By the proof of Lemma 2.2.18 and the definition (2.17),

$$\mathcal{F}_L^p\mathscr{J}_N^k\Omega^\bullet(A\big|_E, V\big|_E) = \bigoplus_{i=\max\{0,p-k\}}^{p} \mathcal{I}_N^{p-i}\Gamma(\wedge^i\mathrm{pr}^*L^{\mathrm{ann}} \otimes \wedge^{\bullet-i}\mathrm{pr}^*L^* \otimes \mathrm{pr}^*V\big|_N).$$

This implies that $\mathcal{F}_L^p\mathscr{J}_N^k\Omega^\bullet(A, V) = 0$ if $p \geq r + 1$. Hence, the filtration is finite and the spectral sequence stabilises as claimed. The rest of the proof is completely analogous the proof of Theorem 2.2.10. ☐

For $L = A\big|_N$, we obtain the following simplifications, analogue to Theorems 2.2.21 and 2.2.24.

Theorem 2.2.26 *Let $N \subseteq M$ be a closed, embedded, and invariant submanifold of $A \Rightarrow M$.*

1. *The first page of the spectral sequence converging to $\mathrm{H}^\bullet(\mathscr{J}_N^k \Omega(A, V))$ is given by*

$$E_1^{p,q} \simeq \begin{cases} \mathrm{H}^{p+q}(A\big|_N, S^p \nu_N^* \otimes V\big|_N) & \text{for } p \leq k \\ 0 & \text{otherwise.} \end{cases}$$

2. *If $A \Rightarrow M$ is linearisable around N, then the spectral sequence stabilises at E_1 and so*

$$\mathrm{H}^\bullet(\mathscr{J}_N^k \Omega^\bullet(A)) \simeq \prod_{j=0}^{k} \mathrm{H}^\bullet(A\big|_N, S^j \nu_N^*).$$

Transverse submanifolds and locality

Another special class of submanifolds is given by transversals. Recall that a **transversal** $\iota \colon X \hookrightarrow M$ is an embedded submanifold, such that the inclusion is transverse to the anchor. Then $\iota^! A = \sharp^{-1}(TX)$ is a Lie subalgebroid. Note that any Lie subalgebroid of A over X is contained in $\iota^! A$. By [BLM16, Theorem 4.1] there exists a tubular neighbourhood $\mathrm{pr} \colon E \to X$ of X over which there is an isomorphism of Lie algebroids

$$A\big|_E \simeq \mathrm{pr}^! \iota^! A. \tag{2.18}$$

This simple local form also manifests itself at the level of cohomology.

Theorem 2.2.27 *Let $\iota \colon X \hookrightarrow M$ be a closed transversal of $A \Rightarrow M$. Then the first page of the Serre spectral sequence associated $\iota^! A$ is given by*

$$E_1^{p,q} \simeq \mathrm{H}^{p+q}(\iota^! A, V\big|_X).$$

Moreover, the spectral sequence stabilises at the first page, and so

$$\mathrm{H}^\bullet(\mathscr{J}_X^\infty \Omega^\bullet(A, V)) \simeq \mathrm{H}^\bullet(\iota^! A, V\big|_X). \tag{2.19}$$

Theorem 2.2.27 follows immediately from the following lemma, which emphasises the local nature of the Serre spectral sequence for formal cohomology.

Lemma 2.2.28 *Let $\iota \colon X \to M$ be a closed transversal of $A \Rightarrow M$, $L \Rightarrow N$ be a Lie subalgebroid over a closed embedded submanifold $N \subseteq X$ and $V \to M$ be a representation of $A \Rightarrow M$. Let $\{E_r^{\bullet,\bullet}\}$ and $\{\hat{E}_r^{\bullet,\bullet}\}$ denote the spectral sequences arising from the inclusions $L \hookrightarrow A$ and $L \hookrightarrow \iota^! A$, respectively. Then the inclusion $j \colon \iota^! A \hookrightarrow A$ induces an isomorphism of spectral sequences*

$$j^* \colon \{E_r^{\bullet,\bullet}\}_{r>0} \xrightarrow{\simeq} \{\hat{E}_r^{\bullet,\bullet}\}_{r>0}.$$

Remark 2.2.29 In fact, a local version of the isomorphism (2.19) in Theorem 2.2.27 holds as well. Namely, for a tubular neighbourhood E of N above which (2.18) holds, we have that

$$\mathrm{H}^\bullet(A\big|_E, V\big|_E) \simeq \mathrm{H}^\bullet(\iota^! A, V\big|_X). \tag{2.20}$$

A way to obtain this isomorphism, which will be used in the proof of Lemma 2.2.28, is by constructing a homotopy operator from an Euler vector field. Denote $B = \iota^! A$, and identify $\mathrm{pr}^! B \simeq A\big|_E$. Let $a \in \Gamma(\ker T\mathrm{pr})$ be the Euler vector field of E, and $\Phi^a_t \colon \mathrm{pr}^! B \to \mathrm{pr}^! B$ be the flow of a, viewed as a section $a \in \Gamma(\mathrm{pr}^! B)$. Then $\mu_t := \Phi^a_{\log t}$ extends to $t = 0$ as $\mu_0 = j \circ \mathrm{pr}^!$, where $j \colon B \hookrightarrow \mathrm{pr}^! B$ is the inclusion of B into $\mathrm{pr}^! B$ over the zero section. Moreover, $\mu_1 = \mathrm{id}_{\mathrm{pr}^! B}$ and $\mu_t\big|_{j(B)} = \mathrm{id}_{j(B)}$. Using parallel transport along the $\mathrm{pr}^! B$-paths μ_t, we obtain a compatible isomorphism of representations $V\big|_E \simeq \mathrm{pr}^!(V\big|_X)$. Analogous to the proof of the Relative Poincaré Lemma [Wei71], for $\omega \in \Omega^\bullet(\mathrm{pr}^! B, \mathrm{pr}^!(V\big|_X))$ one finds

$$\begin{aligned}
\omega - (\mathrm{pr}^!)^* j^* \omega &= \mu_1^* \omega - \mu_0^* \omega \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \mu_t^* \omega \, \mathrm{d}t \\
&= \int_0^1 \frac{1}{t} \mu_t^* \mathscr{L}_a \omega \, \mathrm{d}t \\
&= \int_0^1 \frac{1}{t} \mu_t^* (\mathrm{d}_{\mathrm{pr}^! B} \mathrm{i}_a + \mathrm{i}_a \mathrm{d}_{\mathrm{pr}^! B}) \omega \, \mathrm{d}t \\
&= (\mathrm{d}_{\mathrm{pr}^! B} \circ \mathrm{h} + \mathrm{h} \circ \mathrm{d}_{\mathrm{pr}^! B}) \omega,
\end{aligned}$$

where

$$\mathrm{h}(\omega) = \int_0^1 \frac{1}{t} \mu_t^* \mathrm{i}_a \omega \, \mathrm{d}t.$$

Thus, the map $j^* \colon \Omega^\bullet(\mathrm{pr}^! B, \mathrm{pr}^!(V\big|_X)) \to \Omega^\bullet(B, V\big|_X)$ is a quasi-isomorphism with quasi-inverse $(\mathrm{pr}^!)^*$. Alternatively, one can view μ_t as a retraction of $\mathrm{pr}^! B$ to $j(B)$ and obtain the result that way, see [BP20, Theorem 5.1, Remark 6.7], [Bal12, Theorem 11], or use spectral sequence arguments [Cra00, Theorem 2], see Section 2.3.2 for details.

Proof of Lemma 2.2.28. We can assume that $A = A\big|_E = \mathrm{pr}^! \iota^! A$. First note that j^* and $(\mathrm{pr}^!)^*$ respect the filtrations. In fact, we even have

$$j^* \mathcal{F}^p_L \Omega^\bullet(A, V) = \hat{\mathcal{F}}^p_L \Omega^\bullet(\iota^! A, V\big|_X).$$

Thus, both maps descend to E_0 and we obtain a map between spectral sequences

$$j^* \colon \{E^{\bullet,\bullet}_r\} \to \{\hat{E}^{\bullet,\bullet}_r\}.$$

To show that $j^* \colon E_0 \to \hat{E}_0$ is a quasi-isomorphism, we show that the homotopy h from Remark 2.2.29 is compatible with the filtration on $\Omega^\bullet(A, V)$. Since the Euler vector field vanishes on X we immediately obtain $\mathrm{i}_a \mathcal{F}^p_L \Omega^\bullet(A, V) \subseteq \mathcal{F}^p_L \Omega^{\bullet - 1}(A, V)$. Moreover, the flow μ_t stabilises $\iota^! A$, showing that h indeed respects the filtration and descends to E_0.

Thus, $j^*\colon \{E_1^{\bullet,\bullet}\} \xrightarrow{\simeq} \{\hat{E}_1^{\bullet,\bullet}\}$ is an isomorphism, and by the Mapping Lemma [Wei94, Lemma 5.2.4] the statement follows. □

Coregular submanifolds

A submanifold $N \subseteq M$ is called **coregular submanifold** for a Lie algebroid $A \Rightarrow M$ if

$$W_N := \sharp(A\big|_N) + TN \subseteq TM\big|_N$$

is a subbundle. Then $L = \sharp^{-1}(TN)$ is a Lie subalgebroid, and as for transversals, any Lie subalgebroid of A with base N is also a Lie subalgebroid of L. In fact N is a coregular submanifold if and only if $L = \sharp^{-1}(TN)$ is a subbundle of $A\big|_N$. Coregular submanifolds interpolate between invariant submanifolds ($W_N = TN$) and transverse submanifolds ($W_N = TM\big|_N$). As for these extreme cases, we find that the first page of the Serre spectral sequence associated to L is given by the cohomology of L with values in a classical representation.

Theorem 2.2.30 *Let $N \subseteq M$ be a closed, embedded coregular submanifold of $A \Rightarrow M$ and $V \to M$ a representation. Then $L := \sharp^{-1}(TN) \Rightarrow N$ has a canonical representation on*

$$W_N^{\mathrm{ann}} \subseteq T^*M\big|_N,$$

and the first page of the Serre spectral sequence associated to L is given by

$$E_1^{p,q} \simeq \mathrm{H}^{p+q}(L, S^p(W_N^{\mathrm{ann}}) \otimes V\big|_N).$$

We first describe the representation.

Lemma 2.2.31 *The following defines a representation of $L := \sharp^{-1}(TN) \Rightarrow N$ on W_N^{ann}.*

$$\nabla_b(\mathrm{d}f\big|_N) = \mathrm{d}(\sharp(\tilde{b})(f))\big|_N \tag{2.21}$$

for $\tilde{b} \in \Gamma(A)$ with $\tilde{b}\big|_N = b$ and $f \in \mathcal{I}_N$ such that $\mathrm{d}f\big|_N \in (\sharp A\big|_N)^{\mathrm{ann}}$.

Proof. Since N is coregular, $W_N^{\mathrm{ann}} \to N$ is a vector bundle. Note that, if $f \in \mathcal{I}_N$ satisfies $\mathrm{d}f\big|_N \in (\sharp A\big|_N)^{\mathrm{ann}}$ then $\mathrm{d}f\big|_N \in \Gamma(W_N^{\mathrm{ann}})$. Since N is closed and embedded, all sections of W_N^{ann} can be written in this form.

To see that (2.21) is independent of the choice of the extension $\tilde{b}$, let $b \in \Gamma(L)$ and $\tilde{b}, \hat{b} \in \Gamma(A)$ be two extensions. Then $(\tilde{b} - \hat{b})\big|_N = 0$. Thus, we can write

$$\tilde{b} - \hat{b} = \sum\nolimits_i g_i a_i$$

for suitable $a_i \in \Gamma(A)$ and $g_i \in \mathcal{I}_N$. For $f \in \mathscr{C}^\infty(M)$ such that $\mathrm{d}f\big|_N \in (\sharp A\big|_N)^{\mathrm{ann}}$ we have that

$$\sharp(\tilde{b} - \hat{b})(f) = \sum\nolimits_i g_i \mathrm{d}f(\sharp(a_i))$$

vanishes to second order along N, hence (2.21) does not depend on the choice of $\tilde{b}$.

Next, we show that (2.21) defines a section of W_N^{ann}. For $f \in \mathcal{I}_N$ we have $\sharp(\tilde{b})(f)\big|_N = 0$ as $\sharp(\tilde{b})\big|_N \in \Gamma(TN)$. Thus $\nabla_b(\mathrm{d}f\big|_N) \in \Gamma(TN^{\mathrm{ann}})$. To see that $\nabla_b(\mathrm{d}f\big|_N)$ annihilates $\sharp(A\big|_N)$, let $a \in \Gamma(A)$ be given. Then

$$\begin{aligned}\nabla_b(\mathrm{d}f\big|_N)(\sharp(a)\big|_N) &= \sharp(a)(\sharp(\tilde{b})f)\big|_N \\ &= \underbrace{\sharp([a,\tilde{b}])f\big|_N}_{=0} + \sharp(\tilde{b})(\underbrace{\sharp(a)f}_{\in\mathcal{I}_N})\big|_N = 0.\end{aligned}$$

In conclusion, (2.21) is well-defined. Clearly, (2.21) gives an L-connection on W_N^{ann}, and flatness follows from the fact that, for $b_1, b_2 \in \Gamma(L)$, $[\tilde{b_1}, \tilde{b_2}] \in \Gamma(A)$ extends $[b_1, b_2]$. □

To prove Theorem 2.2.30 we use the fact that any coregular submanifold $N \subseteq M$ is contained in some **minimal transversal** $X \subseteq M$, i.e. one satisfying (see [FM23, Section 9.2])

$$TX\big|_N \cap W_N = TN, \qquad TX\big|_N + W_N = TM\big|_N.$$

Then $N \subseteq X$ is an invariant submanifold for $\iota_X^! A \Rightarrow X$, and we have $L = (\iota_X^! A)\big|_N$. A direct consequence of Lemma 2.2.28 and Lemma 2.2.21 is the following observation.

Lemma 2.2.32 *Let $A \Rightarrow M$ be a Lie algebroid and $V \to M$ a representation of A. Let $N \subseteq M$ be a closed, embedded coregular submanifold, $L = \sharp^{-1}(TN)$ and $\iota_X \colon X \to M$ a minimal transversal containing N. Then the Serre spectral sequences arising from the inclusions $L \hookrightarrow A$ and $L \hookrightarrow \iota_X^! A$ are isomorphic on all pages $r > 0$. In particular,*

$$E_1^{p,q} \simeq \mathrm{H}^{p+q}(L, S^p \nu_N(X)^* \otimes V\big|_N).$$

Proof of Theorem 2.2.30. By Lemma 2.2.32 we have

$$E_1^{p,q} \simeq \mathrm{H}^{p+q}(L, S^p \nu_N(X)^* \otimes V\big|_N)$$

for a minimal transversal $X \hookrightarrow M$ of N. There is a canonical isomorphism $\nu_N(X)^* \simeq W_N^{\mathrm{ann}}$ induced by the inclusion $X \hookrightarrow M$, under which the respective representations of L coincide. Thus, Theorem 2.2.30 follows. □

The main tool in proving the splitting theorem for Lie algebroid transversals in [BLM16] are Euler-like sections. In the following theorem we use a more general version of these sections. Such sections appear for example when blowing up a transversal of codimension one (called an **elementary modification** in [GL13]), see [Sch24, Lemma 5.8]). We obtain the following generalisations (with trivial coefficients) to Remark 2.2.29, Lemma 2.2.28 and Theorem 2.2.27.

Theorem 2.2.33 *Let $N \subseteq M$ be a closed, embedded coregular submanifold of $A \Rightarrow M$ and $B = \sharp^{-1}(TN)$. Suppose there exists a section $a \in \Gamma(A)$ such that $\sharp a \in \Gamma(TM)$ is Euler-like along N (i.e. $\sharp a$ is the Euler vector field of a tubular neighbourhood E of N) and the inner derivation*

$$[a\big|_N, \cdot\,]_B : \Gamma(B) \to \Gamma(B) \tag{2.22}$$

vanishes identically.

1. *The inclusion* $j\colon B \hookrightarrow A\big|_E$ *induces an isomorphism in Lie algebroid cohomology*
$$j^* : \mathrm{H}^\bullet(A\big|_E) \xrightarrow{\sim} \mathrm{H}^\bullet(B).$$

2. *Let* $L \Rightarrow Y$ *be a Lie subalgebroid of* A *with* $Y \subseteq N$*, such that* $a\big|_Y \in \Gamma(L)$*. Then, denoting the spectral sequences arising from the inclusions* $L \hookrightarrow A$ *and* $L \hookrightarrow B$ *by* $\{E_r^{\bullet,\bullet}\}$ *and* $\{\hat{E}_r^{\bullet,\bullet}\}$ *respectively, the map*
$$j^*\colon \{E_r^{\bullet,\bullet}\} \to \{\hat{E}_r^{\bullet,\bullet}\}$$
induced by the inclusion of B *is a quasi-isomorphism on page zero and an isomorphism on all pages* $r > 0$*.*

3. *We have*
$$j^*\colon \mathrm{H}^\bullet(\mathscr{J}_N^\infty \Omega^\bullet(A)) \xrightarrow{\sim} \mathrm{H}^\bullet(B).$$

Proof. First note that $\sharp a\big|_N = 0$ implies $a\big|_N \in \Gamma(B)$, thus (2.22) is well-defined. The proof of the first part is completely analogous to Remark 2.2.29 using the flow of a. The assumption (2.22) ensures that $\mu_t\big|_{j(B)} = \mathrm{id}_{j(B)}$, and μ_0 when considered a map onto its image (note that $\mathrm{im}\mu_0 = B$) replaces the map $\mathrm{pr}^!$. For the second part, following the proof of Lemma 2.2.28 we only need to show that the homotopy operator still respects the filtration. The flow μ_t^* does because $\mu_t\big|_{j(B)} = \mathrm{id}_{j(B)}$, i.e. μ_t stabilises L. Moreover, since by assumption $a\big|_Y \in \Gamma(L)$, we have $\mathrm{i}_a \mathcal{F}_L^p \Omega^\bullet(A) \subseteq \mathcal{F}_L^p \Omega^{\bullet-1}(A)$. In conclusion, the homotopy operator h descends to E_0, showing that $j^*\colon E_0^{\bullet,\bullet} \to \hat{E}_0^{\bullet,\bullet}$ is indeed a quasi-isomorphism. The Mapping Lemma [Wei94, Lemma 5.2.4] then implies the second part. The last part follows from 2 with $L = B$. ☐

2.3 Lie algebroid extensions

In this section, we consider the spectral sequence corresponding to the kernel of a Lie algebroid submersion $\Pi\colon A \to B$. More precisely, we fix a diagram of Lie algebroid maps

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, Rightarrow] & A \arrow[r, "\Pi"] \arrow[d, Rightarrow] & B \arrow[r] \arrow[d, Rightarrow] & 0 \\
 & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q &
\end{tikzcd} \tag{2.23}$$

which is exact, i.e. for each $x \in M$, we have a short exact sequence of vector spaces
$$0 \to L_x \to A_x \to B_{\pi(x)} \to 0,$$
and the base map $\pi\colon M \to Q$ is a surjective submersion. Fix also a representation V of A.

In the case when A and B are over the same base, i.e. $M = Q$ and $\pi = \mathrm{id}_M$, the Serre spectral sequence associated to the $L \subseteq A$ has been studied extensively in [Mac05, Section 7]. For certain proofs, we will use this reference.

Also over the same base, this spectral sequence was studied in great detail in the holomorphic setting in [BMRT15] and in more algebraic setting in [Bru17].

The Serre spectral sequence in the full generality of (2.23) was first considered in [Bra10, Section 3]. Using the notion of a generalised representation defined in Section 1.1.2, in the next theorem we make rigorous the interpretation from [Bra10, Section 3] of the first two pages.

Theorem 2.3.1 *The Serre spectral sequence associated to* $i\colon L \to A$ *satisfies*

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \mathrm{H}^q(L, V)).$$

More precisely, the following hold.

(a) We have a canonical isomorphism (see also (1.13)*)*

$$E_0^{p,q} \simeq \Omega^p(B, \Omega^q(L, V)),$$

where the $\mathscr{C}^\infty(Q)$*-module structure on* $\Omega^q(L, V)$ *is induced by the inclusion*

$$\pi^*\colon \mathscr{C}^\infty(Q) \hookrightarrow \mathscr{C}^\infty(M).$$

(b) The differential d_0 *is* $\mathscr{C}^\infty(Q)$*-linear and, under the isomorphism from (a), it becomes*

$$\mathrm{d}_0(\omega)(\beta_1, \dots, \beta_p) = (-1)^p \mathrm{d}_L(\omega(\beta_1, \dots, \beta_p)),$$

for all $\beta_1, \dots, \beta_p \in \Gamma(B)$*.*

(c) For the induced $\mathscr{C}^\infty(Q)$*-module structure on* $\mathrm{H}^q(L, V)$*, we have isomorphisms*

$$E_1^{p,q} \simeq \Omega^p(B, \mathrm{H}^q(L, V)).$$

(d) There is a generalised representation of B *on the* $\mathscr{C}^\infty(Q)$*-module* $\mathrm{H}^q(L, V)$

$$\nabla\colon \Gamma(B) \times \mathrm{H}^q(L, V) \to \mathrm{H}^q(L, V).$$

(e) Under the isomorphism from (c), $\mathrm{d}_1\colon E_1^{p,q} \to E_1^{p+1,q}$ *corresponds to the differential calculating cohomology of* B *with values in the generalised representation from (d).*

Proof. The first identification follows from the short exact sequence

$$0 \to \mathcal{F}_L^{p+1}\Omega^{p+q}(A, V) \to \mathcal{F}_L^{p}\Omega^{p+q}(A, V) \overset{\mathrm{pr}}{\to} \Omega^p(B, \Omega^q(L, V)) \to 0,$$

where the map pr acts as

$$\mathrm{pr}(\omega)(\beta_1, \dots, \beta_p)(\lambda_1, \dots, \lambda_q) := \omega(\tilde{\beta}_1, \dots, \tilde{\beta}_p, \lambda_1, \dots, \lambda_q),$$

for all $\beta_1, \dots, \beta_p \in \Gamma(B)$ and $\lambda_1, \dots, \lambda_q \in \Gamma(L)$. Here, for a section $\beta \in \Gamma(B)$, we have denoted by $\tilde{\beta} \in \Gamma(A)$ any lift of β, i.e. $\Pi \circ \tilde{\beta} = \beta \circ \pi$. That pr is well-defined and that the sequence is exact in the middle follow directly from the description of the filtration given in Lemma 2.2.5. To show surjectivity, and also for later use, we choose a complement of L in A as in the proof of Lemma 2.2.5, $A = L \oplus C$. Note that Π induces a vector bundle isomorphism $C \simeq \pi^* B$. Using these maps, and Lemma 2.5.1 (1) and (3) we obtain isomorphisms

$$\Omega^\bullet(A,V) \simeq \bigoplus_{p+q=\bullet} \Omega^p(\pi^* B) \otimes_{\mathscr{C}^\infty(M)} \Omega^q(L,V) \simeq \bigoplus_{p+q=\bullet} \Omega^p(B) \otimes_{\mathscr{C}^\infty(Q)} \Omega^q(L,V). \tag{2.24}$$

Under these isomorphisms, we have that

$$\mathcal{F}_L^p \Omega^{p+q}(A,V) \simeq \bigoplus_{0 \leq i \leq q} \Omega^{p+i}(B) \otimes_{\mathscr{C}^\infty(Q)} \Omega^{q-i}(L,V),$$

and pr becomes the projection onto the first component. This ensures its surjectivity. So, we obtain isomorphisms

$$E_0^{p,q} \simeq \Omega^p(B) \otimes_{\mathscr{C}^\infty(Q)} \Omega^q(L,V). \tag{2.25}$$

Next, we need to identify the differential d_0. First note that for $p = 0$ the map pr is just the pullback map along the Lie algebroid map $i\colon L \to A$, therefore a cochain map, and so we have a short exact sequence of cochain complexes

$$0 \to (\mathcal{F}_L^1 \Omega^\bullet(A,V), \mathrm{d}_A) \to (\Omega^\bullet(A,V), \mathrm{d}_A) \overset{\mathrm{pr}}{\to} (\Omega^\bullet(L,V), \mathrm{d}_L) \to 0.$$

This implies that the operator d_0 on $E_0^{0,\bullet}$ corresponds to d_L.

From the diagram (2.23) it follows that the anchor map of L maps to $\ker T\pi$. This implies that the Lie bracket on $\Gamma(L)$ and the representation on $\Gamma(V)$ are $\mathscr{C}^\infty(Q)$-linear. Therefore, the differential d_L on $\Omega^\bullet(L,V)$ is indeed $\mathscr{C}^\infty(Q)$-linear.

Next, let us note that the isomorphism from (2.24) is $\Omega(B)$-linear, where the multiplication on the right is the obvious one, and the one on the left uses the map $\Pi : A \to B$

$$\omega \cdot \eta := \Pi^*(\omega) \wedge \eta,$$

for $\omega \in \Omega(B)$ and $\eta \in \Omega(A,V)$. Moreover, since Π is a Lie algebroid map, it follows that the $\Omega(B)$-module structure is compatible with the differentials

$$\mathrm{d}_A(\omega \cdot \eta) = \mathrm{d}_B(\omega) \cdot \eta + (-1)^p \omega \cdot \mathrm{d}_A(\eta), \tag{2.26}$$

where $\omega \in \Omega^p(B)$. This fact and the description of the filtration imply that the differential d_0 on E_0 is $\Omega^p(B)$-linear in the following sense:

$$\mathrm{d}_0(\omega \cdot \eta) = (-1)^p \omega \cdot \mathrm{d}_0(\eta).$$

This implies that, under the isomorphism (2.25), d_0 becomes:

$$(-1)^p \mathrm{id} \otimes \mathrm{d}_L : \Omega^p(B) \otimes_{\mathscr{C}^\infty(Q)} \Omega^q(L,V) \to \Omega^p(B) \otimes_{\mathscr{C}^\infty(Q)} \Omega^{q+1}(L,V),$$

which is equivalent to the formula given in (b).

Item (c) follows from Lemma 2.5.1 (4)

$$E_1^{p,q} \simeq \mathrm{H}^q(\Omega^p(B) \otimes_{\mathscr{C}^\infty(Q)} \Omega^\bullet(L,V), \mathrm{id} \otimes \mathrm{d}_L) \simeq \Omega^p(B) \otimes_{\mathscr{C}^\infty(Q)} \mathrm{H}^q(L,V).$$

We start by calculating $\mathrm{d}_1 : E_1^{0,q} \to E_1^{1,q}$. Let $c \in \mathrm{H}^q(L,V)$ with representative a closed q-from $\eta \in \Omega^q(L,V)$. Let $\tilde{\eta} \in \Omega^q(A,V)$ be an extension of η. Then we have that

$$\mathrm{d}_A\tilde{\eta} \in \mathcal{F}^1\Omega^{q+1}(A,V),$$

and $\mathrm{d}_1 c \in \Omega^1(B,\mathrm{H}^q(L,V))$ can be calculated as

$$\mathrm{d}_1 c(\beta) = [i^*(\mathrm{i}_{\tilde{\beta}}\mathrm{d}_A\tilde{\eta})] \in \mathrm{H}^q(L,V), \tag{2.27}$$

where $\tilde{\beta} \in \Gamma(A)$ is a lift of $\beta \in \Gamma(B)$, $i^* \colon \Omega^\bullet(A,V) \to \Omega^\bullet(L,V)$ is the pullback, and where we note that $i^*(\mathrm{i}_{\tilde{\beta}}\mathrm{d}_A\tilde{\eta})$ is d_L-closed, and so it defines a cohomology class in $\mathrm{H}^q(L,V)$. We will exploit the fact that this operation is indeed well-defined, i.e. independent of the choices of extension $\tilde{\eta}$ and lift $\tilde{\beta}$. Define the operator from item (d) via the formula

$$\nabla \colon \Gamma(B) \times \mathrm{H}^q(L,V) \to \mathrm{H}^q(L,V), \qquad \nabla_\beta c := \mathrm{d}_1 c(\beta).$$

It is easy to see that the operator is $\mathscr{C}^\infty(Q)$-linear in β. For the other component, to simplify the computation, choose $\tilde{\eta}$ and $\tilde{\beta}$ such that $\mathrm{i}_{\tilde{\beta}}\tilde{\eta} = 0$. Then, for $f \in \mathscr{C}^\infty(Q)$, we have that

$$\begin{aligned}\mathrm{i}_{\tilde{\beta}}\mathrm{d}_A(\widetilde{f\eta}) =& \mathrm{i}_{\tilde{\beta}}\mathrm{d}_A(\pi^*(f)\tilde{\eta}) = \mathrm{i}_{\tilde{\beta}}(\pi^*(\mathrm{d}_B f) \wedge \tilde{\eta}) + \mathrm{i}_{\tilde{\beta}}(\pi^*(f)\mathrm{d}_A\tilde{\eta})\\ =& \pi^*(\mathscr{L}_{\sharp\beta} f)\tilde{\eta} + \pi^*(f)\mathrm{i}_{\tilde{\beta}}\mathrm{d}_A\tilde{\eta},\end{aligned}$$

which yields the second condition in (1.7). To show (1.8), note that we can also write

$$\nabla_\beta c = [i^*(\mathscr{L}_{\tilde{\beta}}\tilde{\eta})], \qquad \text{where} \qquad \mathscr{L}_{\tilde{\beta}} = \mathrm{i}_{\tilde{\beta}} \circ \mathrm{d}_A + \mathrm{d}_A \circ \mathrm{i}_{\tilde{\beta}}.$$

The commutator formula

$$\mathscr{L}_{\tilde{\beta}_1} \circ \mathscr{L}_{\tilde{\beta}_2} - \mathscr{L}_{\tilde{\beta}_2} \circ \mathscr{L}_{\tilde{\beta}_1} = \mathscr{L}_{[\tilde{\beta}_1,\tilde{\beta}_2]},$$

and the fact that $[\tilde{\beta}_1, \tilde{\beta}_2]$ is a lift of $[\beta_1, \beta_2]$ yield now (1.8). Thus, ∇ defines a generalised representation of B on $\mathrm{H}^q(L,V)$.

To obtain (e), we need to show that the map corresponding to $\mathrm{d}_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ under the isomorphism from (c) coincides with the differential calculating Lie algebroid cohomology

$$\mathrm{d}_B \colon \Omega^p(B,\mathrm{H}^q(L,V)) \to \Omega^{p+1}(B,\mathrm{H}^q(L,V)).$$

By the definition of ∇, this holds in degree $p = 0$. Next, one can easily show that both operators satisfy the derivation rule with respect to $\Omega(B)$ (2.26). By also using Lemma 2.5.1, these two properties imply that the differentials must coincide in all degrees $p \geq 0$. □

Remark 2.3.2 In the setting of Theorem 2.3.1, a natural question is whether the generalised representation $\mathrm{H}^q(L,V)$ comes from a classical representation. A candidate for the vector bundle is obtained as follows. First note that, because the anchor of L is tangent to the fibres of π, for any $x \in Q$, $L_x :=$

$L\big|_{\pi^{-1}(x)}$ is a Lie subalgebroid of L and $V_x := V\big|_{\pi^{-1}(x)}$ is endowed with the pullback representation. Consider the collection of vector spaces $\mathcal{H}^\bullet(L,V) \to Q$,

$$\mathcal{H}^\bullet(L,V) := \bigsqcup_{x\in Q} \mathrm{H}^\bullet(L_x, V_x).$$

In some special, but interesting cases, which will be discussed in the sequel, $\mathcal{H}^\bullet(L,V) \to Q$ is a smooth (finite dimensional) vector bundle, and we have an isomorphism

$$\mathrm{H}^\bullet(L,V) \simeq \Gamma(\mathcal{H}^\bullet(L,V)),$$

given as follows: to $c \in \mathrm{H}^\bullet(L,V)$, we assign the section

$$Q \ni x \mapsto \iota_x^* c \in \mathrm{H}^\bullet(L_x, V_x),$$

where $\iota_x \colon L_x \to L$ denotes the inclusion.

These properties might fail to hold for various reasons: the fibres $\mathrm{H}^\bullet(L_x, V_x)$ are infinite dimensional, or their dimension varies with the point, or not every element in $\mathrm{H}^\bullet(L_x, V_x)$ can be extended to an element in $\mathrm{H}^\bullet(L,V)$, etc. However, when the properties do hold, then Theorem (2.3.1) implies that $\mathcal{H}^\bullet(L,V)$ is a classical representation of B, and that

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \mathcal{H}^q(L,V)).$$

Example 2.3.3 Let $\mathfrak{g}$ be a Lie algebra, V be a representation of $\mathfrak{g}$, and $\mathfrak{h} \subseteq \mathfrak{g}$ an ideal. As in [HS53], the second page of the Serre spectral sequence of the inclusion $\mathfrak{h} \subseteq \mathfrak{g}$ is

$$E_2^{p,q} \simeq \mathrm{H}^p(\mathfrak{g}/\mathfrak{h}, \mathrm{H}^q(\mathfrak{h}, V)).$$

Example 2.3.4 Let $A \Rightarrow M$ be a transitive Lie algebroid, which means that the anchor $\sharp \colon A \to TM$ is surjective. Then we have a short exact sequence of Lie algebroids over M

$$0 \to L \to A \to TM \to 0,$$

where $L := \ker \sharp$ is called the isotropy bundle of A. The corresponding Serre spectral sequence is discussed in detail in [Mac05, Section 7]. We have that $L \Rightarrow M$ is a locally trivial bundle of Lie algebras. Moreover, for any representation V of A, we can locally trivialise L and make its action on V constant at the same time. Using this, one obtains the setting of Remark 2.3.2, i.e. $\mathcal{H}^\bullet(L,V) \to M$ is a vector bundle with a flat connection, and there is a canonical isomorphism $\Gamma(\mathcal{H}^\bullet(L,V)) \simeq \mathrm{H}^\bullet(L,V)$. Therefore, the second page of the spectral sequence contains the cohomology of M with coefficients in this flat bundle [Mac05, Theorem 7.4.5]

$$E_2^{p,q} \simeq \mathrm{H}^p(M, \mathcal{H}^q(L,V)).$$

2.3.1 The Leray-Serre spectral sequence

A surjective submersion $\pi\colon M \to Q$ yields a short exact sequence of Lie algebroids

$$\begin{tikzcd}
0 \arrow[r] & \ker T\pi \arrow[r, "i"] \arrow[d, Rightarrow] & TM \arrow[r, "T\pi"] \arrow[d, Rightarrow] & TQ \arrow[r] \arrow[d, Rightarrow] & 0 \\
 & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q &
\end{tikzcd} \tag{2.28}$$

As remarked in Example 2.2.8, the spectral sequence associated to the subalgebroid $\ker T\pi$ of TM is the classical Leray–Serre spectral sequence in de Rham cohomology, which was worked out for example in [Hat60]. We will discuss this construction here in detail, and in the following two subsections, we will discuss two different extensions in the setting of Lie algebroids.

The cohomology $\mathrm{H}^\bullet(\ker T\pi)$ is the foliated cohomology of the foliation on M induced by π. The bundle $\mathcal{H}^q(\ker T\pi) \to Q$ has as fibres the de Rham cohomology of the fibres of π

$$\mathcal{H}^q(\ker T\pi)_x = \mathrm{H}^q(\pi^{-1}(x)).$$

Under appropriate extra conditions, the properties from Remark 2.3.2 hold in this setting.

Theorem 2.3.5 *Assume that $\pi\colon M \to Q$ is a locally trivial fibre bundle with typical fibre a manifold F. If $\mathrm{H}^q(F)$ is finite dimensional, then $\mathcal{H}^q(\ker T\pi)$ is a smooth vector bundle with*

$$\mathrm{H}^q(\ker T\pi) \simeq \Gamma(\mathcal{H}^q(\ker T\pi)). \tag{2.29}$$

Therefore, the second page of the Serre spectral sequence of $\ker T\pi$ is isomorphic to the cohomology of Q with twisted coefficients in the flat bundle $\mathcal{H}^q(\ker T\pi) \to Q$

$$E_2^{p,q} \simeq \mathrm{H}^p(Q, \mathcal{H}^q(\ker T\pi)).$$

To prove Theorem 2.3.5, we first show that $\mathcal{H}^q(\ker T\pi)$ has a smooth vector bundle structure which carries the so-called Gauss-Manin flat connection. Then we prove that (2.29) holds and that the generalised representation is induced by the Gauss-Manin connection.

Lemma 2.3.6 *Let $\pi\colon M \to Q$ be a locally trivial fibre bundle with typical fibre F.*

1. *$\mathcal{H}^q(\ker T\pi) \to Q$ is a locally trivial bundle of vector spaces. There exist canonical local trivialisations with locally constant transition functions.*
2. *If $\mathrm{H}^q(F)$ is finite dimensional, $\mathcal{H}^q(\ker T\pi)$ is a smooth vector bundle endowed with a flat connection.*

Proof. One obtains local trivialisations for $\mathcal{H}^q(\ker T\pi)$ as follows. Any local trivialisation $\lambda_U\colon F \times U \xrightarrow{\sim} \pi^{-1}(U)$ induces a local trivialisation

$$\lambda_U^*\colon \mathcal{H}^q(\ker T\pi)\big|_U \xrightarrow{\sim} \mathrm{H}^q(F) \times U. \tag{2.30}$$

Two trivialisations $\lambda_U \colon F \times U \xrightarrow{\simeq} \pi^{-1}(U)$ and $\lambda_{U'} \colon F \times U' \xrightarrow{\simeq} \pi^{-1}(U')$ are related by a smooth family of diffeomorphisms on the overlap $\delta_{U,U'} \colon U \cap U' \to \mathrm{Diff}(F)$, which yields transition maps in cohomology

$$\delta^*_{U,U'} \colon \mathrm{H}^q(F) \times U \cap U' \xrightarrow{\simeq} \mathrm{H}^q(F) \times U \cap U', \qquad (c, x) \mapsto (\delta_{U,U'}(x)^*(c), x).$$

Since isotopic diffeomorphisms induce the same map in cohomology, it follows that the transition map $\delta^*_{U,U'}$ is locally constant, giving rise to a flat connection on $\mathcal{H}^q(\ker T\pi)$. If $\mathrm{H}^q(F)$ is finite dimensional, one obtains a smooth flat vector bundle. □

Definition 2.3.7 The flat connection from Lemma 2.3.6 is called the **Gauss-Manin connection**. More precisely, its flat sections are locally constant in the trivialisations (2.30).

Lemma 2.3.8 *Let* $\pi \colon M \to Q$ *be a locally trivial fibre bundle with typical fibre* F. *If* $\mathrm{H}^q(F)$ *is finite dimensional, the assignment that sends* $[\omega] \in \mathrm{H}^q(\ker T\pi)$ *to the section of* $\mathcal{H}^q(\ker T\pi)$,

$$Q \ni x \mapsto [\iota_x^* \omega] \in \mathrm{H}^q(\pi^{-1}(x)), \tag{2.31}$$

where $\iota_x \colon \pi^{-1}(x) \to M$ *is the inclusion, is an isomorphism of* $\mathscr{C}^\infty(Q)$*-modules*

$$\mathrm{H}^q(\ker T\pi) \simeq \Gamma(\mathcal{H}^q(\ker T\pi)).$$

Under this identification, the generalised connection on $\mathrm{H}^q(\ker T\pi)$ *from Theorem 2.3.1 (d) corresponds to the Gauss-Manin connection.*

Proof. Note that (2.31) gives a set-theoretic section of the vector bundle $\mathcal{H}^q(\ker T\pi)$, and that the assignment is compatible with the $\mathscr{C}^\infty(Q)$-module structure.

To show that (2.31) is an isomorphism, we follow the arguments of [CM13, Lemma 3]. Fix a basis $\{e_i\}$ of $\mathrm{H}_q(F)$, with dual basis $\{e^i\}$ of $\mathrm{H}^q(F) \simeq \mathrm{H}_q(F)^*$. Using a trivialisation $\pi^{-1}(U) \simeq F \times U \to U$ we obtain the flat local frame $\{\underline{e}^i\}$ for $\mathcal{H}^q(\ker T\pi)\big|_U \simeq \mathrm{H}^q(F) \times U \to U$. Then, for $[\omega] \in \mathrm{H}^q(\ker T\pi)$, the coefficients of the section (2.31) in this frame are given by integrating ω over representatives σ_i of e_i, i.e. $x \mapsto \int_{\sigma_i} \iota_x^* \omega$. Smoothness of ω implies that these coefficients are smooth. Thus, (2.31) is well-defined.

Local injectivity follows from [CM13, Corollary 2], which shows that if $[\iota_x^* \omega] = 0$ for all $x \in U$, then there exists a smooth family of primitives for $\iota_x^* \omega$, i.e. $[\omega] = 0$ is exact. To go global, note that we can glue local primitives using a partition of unity on Q subordinate to local trivialisations.

Similarly, it suffices to show local surjectivity. Note that constant sections are in the image of the map (2.31), because they are obtained via pulling back by the Lie algebroid map $T\mathrm{pr} \colon \ker T\pi\big|_U \simeq TF \times U \to TF$. These sections generate everything. Also note that the classes obtained by the pullback along $T\mathrm{pr}$ are in fact restrictions of de Rham classes on $\pi^{-1}(U)$, and therefore they are flat elements of $\mathrm{H}^q(\ker T\pi\big|_{\pi^{-1}(U)})$. Therefore, the two connections coincide. □

Remark 2.3.9 In the case when the fibres of the locally trivial fibration $\pi\colon M \to Q$ do not have finite dimensional cohomology, note that Lemma 2.3.8 still provides local trivialisations for $\mathcal{H}^q(\ker T\pi) \to Q$ with locally constant transition maps. A direct extension of Theorem 2.3.5 to this setting would require a notion of smooth sections of this infinite dimensional bundle. Instead, we explain here a different approach, based on the homology bundle

$$\mathcal{H}_q(\ker T\pi) \to Q, \qquad \mathcal{H}_q(\ker T\pi)_x := \mathrm{H}_q(\pi^{-1}(x), \mathbb{Z}).$$

As in Lemma 2.3.6, we can build local trivialisations

$$\mathcal{H}_q(\ker T\pi)\big|_U \simeq \mathrm{H}_q(F, \mathbb{Z}) \times U,$$

for which the transition maps come from pushforwards along diffeomorphisms, hence are locally constant group automorphisms. Therefore, $\mathcal{H}_q(\ker T\pi)$ is a smooth, locally trivial bundle of discrete groups over Q. Consider the space of smooth 1-cocycles on $\mathcal{H}_q(\ker T\pi)$

$$\begin{aligned} Z^1(\mathcal{H}_q(\ker T\pi)) = \big\{\varphi \in \mathscr{C}^\infty(\mathcal{H}_q(\ker T\pi)) : \varphi(c_1 + c_2) = \varphi(c_1) + \varphi(c_2),& \\ c_i \in \mathrm{H}_q(\pi^{-1}(x), \mathbb{Z})\big\}& \end{aligned}$$

We have the following version of Theorem 2.3.5.

Theorem 2.3.10 *Pullback to fibres followed by integration yields an isomorphism*

$$\mathrm{H}^q(\ker T\pi) \simeq Z^1(\mathcal{H}_q(\ker T\pi)), \qquad [\omega] \mapsto \Big(\mathrm{H}_q(\pi^{-1}(x), \mathbb{Z}) \ni c \mapsto \int_c \iota_x^* \omega\Big).$$

The proof follows the same lines as that of Lemma 2.3.8 and is also based on [CM13, Lemma 3], which gives the result for a local trivialisation $F \times U \to U$. Namely, the cited result shows that pullback followed by integration gives an isomorphism

$$\mathrm{H}^q(TF \times U) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_q(F, \mathbb{Z}), \mathscr{C}^\infty(U)),$$

and clearly, the second set can be regarded as $Z^1(\mathrm{H}_q(F, \mathbb{Z}) \times U)$.

2.3.2 Pullback Lie algebroids

The Leray-Serre spectral sequence can be generalised to the following setting. Consider the pullback Lie algebroid $\pi^! B \Rightarrow M$ of a Lie algebroid $B \Rightarrow Q$ along a surjective submersion $\pi : M \to Q$ (recall (1.14)). This fits into a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker T\pi \arrow[r, "i"] \arrow[d, Rightarrow] & \pi^! B \arrow[r, "\pi^!"] \arrow[d, Rightarrow] & B \arrow[r] \arrow[d, Rightarrow] & 0 \\ & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q & \end{tikzcd} \tag{2.32}$$

Using Theorems 2.3.1 and 2.3.5 we obtain the following.

Theorem 2.3.11 *Let $B \Rightarrow Q$ be a Lie algebroid over the base of a fibre bundle $\pi\colon M \to Q$ with typical fibre F. Assume that $\mathrm{H}^\bullet(F)$ is finite dimensional. The Serre spectral sequence associated to $\ker T\pi \subseteq \pi^! B$ converges to $\mathrm{H}^\bullet(\pi^! B)$ and satisfies*

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \mathcal{H}^q(\ker T\pi)),$$

where the representation of B on $\mathcal{H}^q(\ker T\pi)$ is the Gauss-Manin connection factored through the anchor, i.e. $\nabla_b = \nabla^{GM}_{\sharp b}$.

Proof. Using Theorem 2.3.5 all that is left to show is the statement about the representation of B. We argue locally; $\pi^{-1}(U) \simeq F \times U \overset{\mathrm{pr}}{\to} U$. If $\theta \in \Omega^\bullet(F)$ is closed, then $[\mathrm{pr}^*\theta]$ is a flat section of $\mathcal{H}^\bullet(\ker T\pi)\big|_U$ for the Gauss-Manin connection, and so also flat for the B-connection $\nabla^{GM}_\sharp$. On the other hand, note that

$$\tilde{\eta} := (\sharp_{\pi^! B})^* \mathrm{pr}^*\theta \in \Omega^\bullet(\pi^! B\big|_{\pi^{-1}(U)})$$

satisfies $d_{\pi^! B}\tilde{\eta} = 0$. Therefore, the definition of the representation of B from (2.27) implies that $[\mathrm{pr}^*\theta]$ is flat also with respect to ∇. Using that sections of the form $[\mathrm{pr}^*\theta]$ span all sections of $\mathcal{H}^\bullet(\ker T\pi)\big|_U$ and the Leibniz rule (1.7), we find that the two B-connections coincide. □

When the cohomology of the typical fibre is fairly simple, one can get more precise results. The following was obtained in [Cra00, Theorem 2], using the same spectral sequence.

Corollary 2.3.12 *Let $B \Rightarrow Q$ be a Lie algebroid and $\pi\colon M \to Q$ fibre bundle such that the typical fibre F is k-connected, i.e. has cohomology*

$$\mathrm{H}^q(F) = \begin{cases} \mathbb{R} & \text{if } q = 0 \\ 0 & \text{if } 1 \leq q \leq k. \end{cases}$$

Then the map

$$(\pi^!)^*\colon \mathrm{H}^q(B) \to \mathrm{H}^q(\pi^! B)$$

is an isomorphism for $q = 0, \dots, k$ and is injective for $q = k+1$.

Example 2.3.13 Another class of fibres with relatively simple cohomology are spheres $F = S^n$. In this case, we obtain a Lie algebroid version of the classical Gysin sequence. On the second page of the associated Serre spectral sequence only the zeroth and n-th row are non-trivial, and the only non-trivial differential is on page E_n. One obtains a long exact sequence (see for example the analogous discussion in [BT82], before Proposition 14.33)

$$\ldots \longrightarrow \mathrm{H}^k(\pi^! B) \xrightarrow{a} \mathrm{H}^{k-n}(B, \mathcal{H}^n(\ker T\pi)) \xrightarrow{\mathrm{d}_n} \mathrm{H}^{k+1}(B) \xrightarrow{b} \mathrm{H}^{k+1}(\pi^! B) \longrightarrow \ldots$$

The maps can be described as follows. First, note that we have an isomorphism of flat bundles

$$\mathcal{H}^n(\ker T\pi) \simeq o(M) \times_{\mathbb{Z}_2} \mathbb{R},$$

where $o(M) \to Q$ is the double cover corresponding to the two possible orientations on the fibres of $M \to Q$. Under this identification, a is the map integrating along the fibres, $\mathrm{d}_n = \sharp^* e \wedge$, where $e \in \mathrm{H}^{n+1}(Q)$ is the Euler class of the sphere bundle, and $b = (\pi^!)^*$ is the pullback. See [Sch24] or Section 3.8 for details.

2.3.3 Submersions by Lie algebroids

A different class of Lie algebroid extensions for which the Leray-Serre spectral sequence admits an interesting generalisation are the **submersions by Lie algebroids**, introduced and studied recently in [Fre19]. These are pairs (A, π) composed of a Lie algebroid $A \Rightarrow M$ and a surjective submersion $\pi\colon M \to Q$ such that $T\pi \circ \sharp$ is surjective. Hence, if $L := \ker(T\pi \circ \sharp)$, there is a short exact sequence of Lie algebroids

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, Rightarrow] & A \arrow[r, "T\pi \circ \sharp"] \arrow[d, Rightarrow] & TQ \arrow[r] \arrow[d, Rightarrow] & 0 \\
& M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q &
\end{tikzcd} \tag{2.33}$$

For locally trivial submersion by Lie algebroids (recalled below), a Leray-type spectral sequence for $\mathrm{H}^\bullet(A, V)$ was constructed in [Fre19, Section 5]. In particular, the second page of this spectral sequence contains the Čech cohomology of Q with values in a certain presheaf. We show that the sheafification of this presheaf appears naturally in our setting of the Serre spectral sequence associated to the extension, and we explain in Theorem 2.3.18 how the two constructions are related.

Note that, for a submersion by Lie algebroids (A, π), the fibres of π are transverse submanifolds for A. By the normal form theorem [BLM16, Theorem 4.1] each fibre $\pi^{-1}(x)$ has a tubular neighbourhood $\mathrm{pr}\colon \mathcal{U} \to \pi^{-1}(x)$, such that there is an isomorphism of Lie algebroids $A\big|_{\mathcal{U}} \simeq \mathrm{pr}^! L_x$ covering $\mathrm{id}_{\mathcal{U}}$, where $L_x := L\big|_{\pi^{-1}(x)}$. If we could take $\mathcal{U}$ to be part of a trivialisation of π, i.e. $\mathcal{U} = \pi^{-1}(U)$ and $(\pi, \mathrm{pr})\colon \pi^{-1}(U) \xrightarrow{\simeq} U \times \pi^{-1}(x)$ is a diffeomorphism (e.g. if π is proper), we would obtain a local trivialisation of A

$$A\big|_{\pi^{-1}(U)} \simeq TU \times L_x. \tag{2.34}$$

If Q can be covered by such local trivialisations, the submersion by Lie algebroids (2.33) is called **locally trivial**. A condition equivalent to local triviality is the existence of **complete Ehresmann connections** [Fre19, Theorem 3]. This is a splitting of vector bundles $A = L \oplus C$ such that the induced horizontal lift

$$\cdot^{\,\mathrm{hor}} : \Gamma(TQ) \to \Gamma(C) \subseteq \Gamma(A)$$

maps complete vector fields to complete sections of A (i.e. whose anchor is complete). The resulting parallel transport can be used to locally trivialise A.

Local trivialisations (2.34) yield local trivialisations of the "vector bundle" $\mathcal{H}^q(L, V) \to Q$ with locally constant transition functions [Fre19, Lemma 4]. However, since the fibres of $\mathcal{H}^q(L, V) \to Q$ will in general be infinite dimensional (e.g. if L is the zero Lie algebroid), the interpretation from Remark 2.3.2 of $\mathrm{H}^\bullet(L, V)$ as sections of a vector bundle is not suitable in this general setting. Instead, we show that elements of $\mathrm{H}^\bullet(L, V)$ that are flat under the representation of TQ constitute a sheaf, which is the sheafification of the presheaf appearing in [Fre19, Section 5]. Using this, we describe the second page of the Serre spectral sequence in terms of cohomology of Q with values in this sheaf.

Lemma 2.3.14 *The assignment, sending an open set $U \subseteq Q$ to*

$$\mathcal{S}^q_{L,V}(U) = \{c \in \mathrm{H}^q(L|_{\pi^{-1}(U)}, V|_{\pi^{-1}(U)}) : \nabla c = 0\},$$

constitutes a sheaf on Q, where ∇ is the generalised representation from Theorem 2.3.1 (d). Moreover, if A is locally trivial, then

$$0 \longrightarrow \mathcal{S}^q_{L,V} \longrightarrow \Omega^0_Q(\,\cdot\,, \mathrm{H}^q(L,V)) \xrightarrow{\mathrm{d}_\nabla} \Omega^1_Q(\,\cdot\,, \mathrm{H}^q(L,V)) \xrightarrow{\mathrm{d}_\nabla} \dots \tag{2.35}$$

is a resolution of $\mathcal{S}^q_{L,V}$ by fine sheaves.

Proof. We first check that the presheaf $\mathcal{S}^q_{L,V}$ is complete.

Let $\{U_i\}_i$ be a family of open subsets of Q and let $U := \bigcup_i U_i$. Consider $c \in \mathcal{S}^q_{L,V}(U)$ such that $c|_{U_i} = 0$, for all i. We want to show that $c = 0$. Let $\omega \in \Omega^q(L|_{\pi^{-1}(U)}, V|_{\pi^{-1}(U)})$ be a representative of c. Then $\omega|_{\pi^{-1}(U_i)} = \mathrm{d}_L\eta_i$, for some $\eta_i \in \Omega^{q-1}(L|_{\pi^{-1}(U_i)}, V|_{\pi^{-1}(U_i)})$. Choose a partition of unity $\{\chi_i\}_i$ on U subordinate to the cover $\{U_i\}_i$. Define the $q-1$-form

$$\eta := \sum_i \pi^*\chi_i\,\eta_i \in \Omega^{q-1}(L|_{\pi^{-1}(U)}, V|_{\pi^{-1}(U)}).$$

Since d_L is $\mathscr{C}^\infty(Q)$-linear, we have that

$$\mathrm{d}_L\eta = \sum_i \pi^*\chi_i\,\mathrm{d}_L\eta_i = \sum_i \pi^*\chi_i\,\omega = \omega,$$

hence indeed $c = 0$.

Next, consider classes $c_i \in \mathcal{S}^q_{L,V}(U_i)$ such that

$$c_i|_{U_i\cap U_j} = c_j|_{U_i\cap U_j}.$$

To glue these elements to a global section over U, pick representatives $\omega_i \in \Omega^q(L|_{\pi^{-1}(U_i)}, V|_{\pi^{-1}(U_i)})$ of c_i. Then we can write

$$\omega_i|_{U_i\cap U_j} - \omega_j|_{U_i\cap U_j} = \mathrm{d}_L\theta_{i,j}, \quad \text{with} \quad \theta_{i,j} \in \Omega^{q-1}(L|_{\pi^{-1}(U_i\cap U_j)}, V|_{\pi^{-1}(U_i\cap U_j)}).$$

Define

$$\omega = \sum_i \pi^*\chi_i\,\omega_i \in \Omega^q(L|_{\pi^{-1}(U)}, V|_{\pi^{-1}(U)}).$$

Using that d_L is $\mathscr{C}^\infty(Q)$-linear, it follows that ω is closed and that

$$\omega|_{U_j} = \sum_i \pi^*\chi_i(\omega_j|_{U_i\cap U_j} + \mathrm{d}_L\theta_{i,j}) = \omega_j + \mathrm{d}_L(\sum_i \pi^*\chi_i\,\theta_{i,j}).$$

Hence, $c := [\omega] \in H^q(L|_{\pi^{-1}(U)}, V|_{\pi^{-1}(U)})$ satisfies $c|_{U_j} = c_j$. To show that $\nabla c = 0$, we use that flatness is a local property. Namely, for any vector field

$X \in \Gamma(TU)$ we have that $\nabla_X c\big|_{U_i} = \nabla_{X\big|_{U_i}} c_i = 0$, which implies, as in the first part, that $\nabla_X c = 0$.

We conclude that $\mathcal{S}^q_{L,V}$ is a sheaf.

Exactness of (2.35) at $\Omega^0_Q(\,\cdot\,, \mathrm{H}^q(L,V))$ is clear. We show exactness in degree $p \geq 1$ and at some point $x \in Q$. First, by the proof of [Fre19, Lemma 4], the fact that A admits a complete connection implies that we can simultaneously trivialise A and the representation

$$(A\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) \simeq (TU \times L_x, \mathrm{pr}_2^\sharp V_x), \tag{2.36}$$

where $V_x := V\big|_{\pi^{-1}(x)}$, U is a small neighbourhood of x, and $\mathrm{pr}_2 \colon U \times \pi^{-1}(x) \to \pi^{-1}(x)$ is the second projection. We shrink U to admit coordinates $\{y_i\}$ in which U corresponds to a ball. Using this isomorphism, we can decompose

$$\Omega^\bullet(A\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) \simeq \bigoplus_{p+q=\bullet} \Omega^p(U, \Omega^q(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x)).$$

A form $\eta \in \Omega^p(U, \Omega^q(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x))$ can be written uniquely as

$$\eta = \frac{1}{p!} \sum_{i_1 \dots i_p} \eta_{i_1 \dots i_p} \mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_p},$$

where the fully skew-symmetric coefficients $\eta_{i_1 \dots i_p} \in \Omega^q(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x)$ can be thought of as smooth families of forms on L_x with values in V_x

$$U \ni y \mapsto \eta_{i_1 \dots i_p}(y) \in \Omega^q(L_x, V_x).$$

Under these identifications, the A-differential decomposes as $\mathrm{d}_A = \mathrm{d}_{L_x} + (-1)^q \mathrm{d}$, where d_{L_x} has bidegree $(0,1)$ and is the differential of L_x and d has bidegree $(1,0)$ and is the 'de Rham' differential on U, i.e.

$$\mathrm{d}_{L_x}\eta = \frac{1}{p!} \sum_{i_1 \dots i_p} (\mathrm{d}_{L_x}\eta_{i_1 \dots i_p}) \mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_p} \qquad \text{and}$$

$$\mathrm{d}\eta = \frac{1}{p!} \sum_{i_0 \dots i_p} \partial_{y_{i_0}}(\eta_{i_1 i_2 \dots i_p}) \mathrm{d}y_{i_0} \wedge \mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_p}.$$

To show exactness of (2.35), let $c = [\eta] \in \Omega^p(U, \mathrm{H}^q(L,V))$ be such that $\mathrm{d}_\nabla c = 0$, where U is some neighbourhood of x. We need to show that, after possibly shrinking U, there is some $e \in \Omega^{p-1}(U, \mathrm{H}^q(L,V))$ such that $c = \mathrm{d}_\nabla e$. So we may assume that there is a trivialisation (2.36) above U and there are coordinates $\{y_i\}$ centred at x in which U corresponds to a ball. Then, under the above identifications, d_∇ becomes the de Rham differential, so

$$0 = \mathrm{d}_\nabla c = \mathrm{d}_\nabla[\eta] = [\mathrm{d}_A \eta] = (-1)^q[\mathrm{d}\eta] \in \Omega^{p+1}(U, \mathrm{H}^q(L,V)).$$

Hence, any coefficient $(\mathrm{d}\eta)_{i_0 i_1 \dots i_p}$ of $\mathrm{d}\eta$ is d_{L_x} exact, so we can write

$$\mathrm{d}\eta = \mathrm{d}_{L_x}\theta, \quad \text{with} \quad \theta \in \Omega^{p+1}(U, \Omega^{q-1}(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x)).$$

Next, consider the standard homotopy operators from the Poincaré Lemma on U, corresponding to the contraction $\mu_t(y) = ty$ in the coordinates $\{y_i\}$, i.e. if $\xi = \sum_i y_i \partial_{y_i}$, let

$$\mathrm{h}\alpha = \int_0^1 \frac{1}{t} \mathrm{i}_\xi \mu_t^* \alpha \, \mathrm{d}t = \int_0^1 \frac{t^{\ell-1}}{(\ell-1)!} \sum_{i_1 \dots i_\ell} y_{i_1} \alpha_{i_1 \dots i_p}(ty) \mathrm{d}y_{i_2} \wedge \dots \wedge \mathrm{d}y_{i_\ell} \, \mathrm{d}t.$$

Note that these operators make sense for $\alpha \in \Omega^\ell(U, \Omega^\bullet(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x))$, $\ell \geq 1$, and still satisfy the homotopy relation and, moreover, commute with d_{L_x}, i.e.

$$\alpha = \mathrm{dh}\alpha + \mathrm{hd}\alpha \quad \text{ and } \quad \mathrm{d}_{L_x} \mathrm{h}\alpha = \mathrm{hd}_{L_x} \alpha.$$

This implies that

$$\eta = \mathrm{dh}\eta + \mathrm{hd}\eta = \mathrm{dh}\eta + \mathrm{hd}_{L_x} \theta = \mathrm{dh}\eta + \mathrm{d}_{L_x} \mathrm{h}\theta,$$

and also that

$$\mathrm{d}_{L_x} \mathrm{h}\eta = \mathrm{hd}_{L_x} \eta = 0.$$

Thus, the element $\mathrm{h}\eta \in \Omega^{p-1}(U, \Omega^q(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x))$ defines a class $e := [\mathrm{h}\eta] \in \Omega^{p-1}(U, \mathrm{H}^q(L, V))$ which satisfies

$$\mathrm{d}_\nabla e = (-1)^q [\mathrm{dh}\eta] = (-1)^q [\eta - \mathrm{d}_{L_x} \mathrm{h}\theta] = (-1)^q [\eta] = (-1)^q c.$$

Therefore c is d_∇-exact.

Finally, the resolution is by fine sheaves because $\Omega^\bullet(U, \mathrm{H}^q(L, V))$ is a $\mathscr{C}^\infty(U)$-module. □

Theorem 2.3.15 *Let $(A \Rightarrow M, \pi \colon M \to Q)$ be a locally trivial submersion by Lie algebroids, and $V \to M$ a representation of A. The spaces on the second page of the associated Serre spectral sequence are isomorphic to the cohomology of Q with coefficients in the sheaf $\mathcal{S}^q_{L,V}$*

$$E_2^{p,q} \simeq \mathrm{H}^p(Q, \mathcal{S}^q_{L,V}).$$

Proof. By Lemma 2.3.14 and [War83, Theorem 5.25] we obtain a canonical isomorphism

$$\mathrm{H}^\bullet(Q, \mathcal{S}^q_{L,V}) \simeq \mathrm{H}^\bullet(Q, \mathrm{H}^q(L, V)),$$

since (2.35) is a resolution by fine sheaves. This together with Theorem 2.3.1 proves the claim. □

Remark 2.3.16 In [Fre19], a different spectral sequence was constructed for a submersion by Lie algebroids (A, π). Namely, as explained in [Fre19, Proposition 3], the Čech-Lie algebroid double complex of an open cover $\mathcal{U} = \{U_i\}$ of Q,

$$C^p(\mathcal{U}, \Omega^q(A, V)) := \prod_{i_0 < \dots < i_p} \Omega^q(A|_{\pi^{-1}(U_{i_0 \dots i_p})}, V|_{\pi^{-1}(U_{i_0 \dots i_p})}), \quad \mathrm{d} = \delta + (-1)^p \mathrm{d}_A, \tag{2.37}$$

is quasi-isomorphic to $(\Omega^\bullet(A,V), \mathrm{d}_A)$, and so the associated spectral sequence $\check{E}_r^{p,q}(\mathcal{U})$ converges to $\mathrm{H}^\bullet(A,V)$. Moreover, the second page is the Čech cohomology of $\mathcal{U}$ with values in the pushforward presheaf $\mathcal{P}^q = \pi_*(\mathrm{H}^q(A,V))$

$$\check{E}_2^{p,q}(\mathcal{U}) \simeq \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{P}^q). \tag{2.38}$$

Assume now that the submersion by Lie algebroids is **locally trivial**. Then, as shown in [Fre19, Lemma 4], $\mathcal{P}^q$ is a locally constant presheaf: if $x \in Q$ and U is a contractible neighbourhood of x, which admits a trivialisation as in (2.36), then the inclusion $L_x \hookrightarrow A$ induces an isomorphism

$$\mathcal{P}^q(U) = \mathrm{H}^q(A\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) \simeq \mathrm{H}^q(TU \times L_x, \mathrm{pr}_2^\sharp(V_x)) \simeq \mathrm{H}^q(L_x, V_x). \tag{2.39}$$

To relate [Fre19, Proposition 3] to our Theorem 2.3.15, we first note the following.

Lemma 2.3.17 *If the submersion by Lie algebroids is locally trivial, then the sheafification of the presheaf $\mathcal{P}^q$ is canonically isomorphic to the sheaf $\mathcal{S}^q_{L,V}$.*

Proof. For every $U \subseteq Q$, the pullback along the inclusion $L \hookrightarrow A$ induces a map

$$\mathrm{H}^q(A\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) \to \mathrm{H}^q(L\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}). \tag{2.40}$$

From the definition of the connection on $\mathrm{H}^q(L,V)$, we see that this map takes values in flat section. Thus, we have a canonical map of presheaves $\mathcal{P}^q \to \mathcal{S}^q_{L,V}$. We show that this is an isomorphism locally, which will imply the claim. Let $x \in Q$. Fix an open subset $U \subseteq Q$ as in the second half of the proof of Lemma 2.3.14, i.e. U corresponds to a ball centred at 0 in the coordinates $\{y_i\}$ and we fix an trivialisation over U as in (2.36). By (2.39), the composition

$$\mathrm{H}^q(A\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) \to \mathrm{H}^q(L\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)}) \to \mathrm{H}^q(L_x, V_x)$$

is an isomorphism, hence the map (2.40) is injective. To show surjectivity, fix $c = [\eta] \in \mathrm{H}^q(L\big|_{\pi^{-1}(U)}, V\big|_{\pi^{-1}(U)})$. As in the proof of Lemma 2.3.14, we regard $\eta \in \Omega^q(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x)$ as a family of forms $\eta(y) \in \Omega^q(L_x, V_x)$ depending smoothly on $y \in U$ and satisfying $\mathrm{d}_{L_x}\eta(y) = 0$. On the other hand, we have that

$$0 = \mathrm{d}_\nabla[\eta] = (-1)^q[\mathrm{d}\eta] = \sum_i [\partial_{y_i}\eta]\mathrm{d}y_i.$$

Thus, we can write $\partial_{y_i}\eta = \mathrm{d}_{L_x}\theta^i(y)$, for some $\theta^i \in \Omega^{q-1}(\mathrm{pr}_2^! L_x, \mathrm{pr}_2^\sharp V_x)$. Therefore,

$$\eta(y) - \eta(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\eta(ty)\mathrm{d}t = \sum_i \int_0^1 y_i \partial_{y_i}\eta(ty)\mathrm{d}t = \mathrm{d}_{L_x}\sum_i \int_0^1 y_i\theta^i(ty)\mathrm{d}t.$$

Hence $c = [\eta(0)]$, which is clearly in the image of the map (2.40). Namely, the trivialisation (2.36) yields a Lie algebroid map $\mathrm{pr}_2 \colon A\big|_{\pi^{-1}(U)} \to L_x$ which respects representations, and the class $[\mathrm{pr}_2^*\eta(0)]$ is mapped under (2.40) to c. ☐

For **good covers** $\mathcal{U}$, we obtain that the second page of the spectral sequence $\check{E}_r^{p,q}(\mathcal{U})$ constructed in [Fre19] is isomorphic to the second page of the Serre spectral sequence $E_r^{p,q}$ corresponding to the extension (2.33)–both converging to $\mathrm{H}^\bullet(A, V)$.

Theorem 2.3.18 *Let $\mathcal{U}$ be a good cover of Q (i.e. all intersections $U_{i_0\dots i_p}$ are contractible), so that over each $U_i \in \mathcal{U}$ a trivialisation as in* (2.36) *exists. There are isomorphisms*

$$\check{E}_2^{p,q}(\mathcal{U}) \simeq \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{P}^q) \simeq \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{S}^q_{L,V}) \simeq \mathrm{H}^p(Q, \mathcal{S}^q_{L,V}) \simeq E_2^{p,q}.$$

Proof. The first isomorphism was observed already in (2.38). The second isomorphism follows from the proof of Lemma 2.3.17, which shows that, for any intersection $U_{i_0\dots i_p}$ of elements in $\mathcal{U}$,

$$\mathcal{P}^q(U_{i_0\dots i_p}) = \mathcal{S}^q_{L,V}(U_{i_0\dots i_p});$$

hence, the Čech complexes computing the two cohomology groups coincide. For the third isomorphism, we use Leray's result [Ler46], which says that a good cover of a manifold can be used to calculate cohomology with values in a locally constant sheaf (see, e.g. [GQ22, Theorem 15.30]). The last isomorphism was the content of Theorem 2.3.15. ☐

Coupling Poisson and Dirac structures

We apply the tools developed in this section to a class of submersions by Lie algebroids coming from Poisson geometry which were introduced by Vorobjev in the study of normal forms for Poisson structures around symplectic leaves. In particular, we recover the description from [VBV18, Theorem 4.8] of the first Poisson cohomology group of coupling Poisson structures.

A **Poisson structure** (see e.g. [Wei83]) on a manifold M is a bivector field $w \in \Gamma(\wedge^2 TM)$ satisfying $[\![w, w]\!]_{\mathrm{S}} = 0$ for the Schouten bracket. A Poisson structure yields a Lie algebroid structure on the cotangent bundle T^*M with anchor and bracket given by

$$\sharp\alpha := w(\alpha, \cdot\,), \qquad [\alpha, \beta]_w := \mathscr{L}_{\sharp\alpha}\beta - \mathrm{i}_{\sharp\beta}\mathrm{d}\alpha, \qquad \alpha, \beta \in \Gamma(T^*M).$$

The **Poisson cohomology** of (M, w) is defined as the Lie algebroid cohomology of $T^*M \Rightarrow M$, and can be computed using the complex of multivector fields and differential $\mathrm{d}_w := [\![w, \cdot\,]\!]_{\mathrm{S}}$, i.e.

$$(\Omega^\bullet(T^*M), \mathrm{d}_{T^*M}) \simeq (\Gamma(\wedge^\bullet TM), \mathrm{d}_w). \tag{2.41}$$

A **Dirac structure** is a subbundle $D \subseteq TM \oplus T^*M$ which is Lagrangian for the canonical, split signature pairing on $TM \oplus T^*M$ and whose space of sections is closed under the Dorfman bracket (for basics on Dirac geometry, see [Cou90, Bur13, CFM21], as well as Section 1.1.3). Any Dirac structure is a Lie algebroid with the Dorfman bracket. Dirac structures generalise simultaneously Poisson structures and closed forms (via their graphs), as well as foliations ($D = T\mathcal{F} \oplus (T\mathcal{F})^{\mathrm{ann}}$).

A Poisson structure w on the total space of a surjective submersion $\pi\colon M \to Q$ is called **horizontally nondegenerate** [Vor01, Vor05] if it satisfies

$$\ker T\pi \oplus \sharp(\ker T\pi)^{\mathrm{ann}} = TM,$$

where $(\ker T\pi)^{\mathrm{ann}} \subseteq T^*M$ is the annihilator of $\ker T\pi$, i.e. the space of 'horizontal' 1-forms.

More generally, a Dirac structure on $D \subseteq TM \oplus T^*M$ on the total space of a surjective submersion $\pi\colon M \to Q$ is called horizontally nondegenerate [Wad08], if it satisfies

$$D \cap \big(\ker T\pi \oplus (\ker T\pi)^{\mathrm{ann}}\big) = 0.$$

As explained in [Vor01, Vor05] for Poisson and in [Wad08] in general, a horizontally nondegenerate Dirac structure D gives rise to the following **coupling data**:

1. A vertical bivector field $\mathcal{W} \in \Gamma(\wedge^2 \ker T\pi)$;
2. An Ehresmann connection, i.e. a subbundle $H \subseteq TM$ such that $TM = H \oplus \ker T\pi$;
3. A horizontal 2-form $\mathbb{F} \in \Gamma(\wedge^2(\ker T\pi)^{\mathrm{ann}})$.

The relation between the Dirac structure and the coupling data is

$$D = \{(X, \mathrm{i}_X\mathbb{F}) : X \in H\} \oplus \{(\mathcal{W}^\sharp\alpha, \alpha) : \alpha \in H^{\mathrm{ann}}\}. \tag{2.42}$$

In fact, [Wad08, Theorem 2.9] shows that this equality yields a one-to-one correspondence between horizontally nondegenerate Dirac structures D and triples $(\mathcal{W}, H, \mathbb{F})$ satisfying the following four conditions (see [Vor05, Proposition 2.4] for the Poisson case, and also [Mat13])

1. $\mathcal{W}$ is a Poisson structure, i.e.
$$[\mathcal{W}, \mathcal{W}] = 0;$$
2. the parallel transport of H preserves $\mathcal{W}$, i.e.
$$\mathscr{L}_{X^{\mathrm{hor}}}\mathcal{W} = 0, \qquad \text{for all } X \in \Gamma(TQ), \tag{2.43}$$
where $X^{\mathrm{hor}} \in \Gamma(H)$ denotes the H-horizontal lift of X;
3. $\mathbb{F}$ is horizontally closed, i.e.
$$\mathrm{d}\mathbb{F}(X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}, X_3^{\mathrm{hor}}) = 0, \qquad \text{for all } X_1, X_2, X_3 \in \Gamma(TQ);$$
4. the curvature of H, defined as
$$R_H(X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}) := [X_1, X_2]^{\mathrm{hor}} - [X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}] \in \Gamma(\ker T\pi),$$
satisfies
$$R_H(X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}}) = \mathcal{W}^\sharp\mathrm{d}(\mathbb{F}(X_1^{\mathrm{hor}}, X_2^{\mathrm{hor}})), \qquad \text{for all } X_1, X_2 \in \Gamma(TQ).$$

The horizontally nondegenerate Dirac structure D is the graph of a Poisson structure w if and only if $\mathbb{F}$ is nondegenerate when regarded as a 2-form on the vector bundle $TM/(\ker T\pi)$.

A horizontally non-degenerate Dirac structure D yields a submersion by Lie algebroids

$$\begin{tikzcd}
0 \arrow[r] & (\ker T\pi)^* \arrow[r, "i"] \arrow[d, Rightarrow] & D \arrow[r, "T\pi\circ\sharp"] \arrow[d, Rightarrow] & TQ \arrow[r] \arrow[d, Rightarrow] & 0 \\
& M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q &
\end{tikzcd} \tag{2.44}$$

Here, we have used the decomposition (2.42) and the following isomorphism

$$\ker T\pi \circ \sharp = \{(\mathcal{W}^\sharp(\alpha), \alpha) : \alpha \in H^{\mathrm{ann}}\} \simeq H^{\mathrm{ann}} \simeq (\ker T\pi)^*.$$

Under this identification, the Lie algebroid structure on $(\ker T\pi)^*$ becomes the family of cotangent Lie algebroids T^*_xM, $x \in Q$, corresponding to the Poisson manifolds $(M_x, \mathcal{W}_x)$, where $M_x = \pi^{-1}(x)$ and $\mathcal{W}_x = \mathcal{W}|_{M_x}$.

Since D is transverse to the Dirac structure $\ker T\pi \oplus (\ker T\pi)^{\mathrm{ann}}$, the pairing of the Courant algebroid $TM \oplus T^*M$ gives a canonical isomorphism $D^* \simeq \ker T\pi \oplus (\ker T\pi)^{\mathrm{ann}}$. This yields a decomposition of the space of forms on D

$$\Omega^\bullet(D) = \bigoplus_{p+q=\bullet} \Omega^p(Q, \Gamma(\wedge^q \ker T\pi)).$$

Under this identification, the differential decomposes as $\mathrm{d}_D = \mathrm{d}^{(0,1)} + \mathrm{d}^{(1,0)} + \mathrm{d}^{(2,-1)}$, where

$$\begin{aligned}
(\mathrm{d}^{(0,1)}\eta)(X_1 \dots X_p) =& (-1)^p [\![\mathcal{W}, \eta(X_1 \dots X_p)]\!]_{\mathrm{S}} \\
(\mathrm{d}^{(1,0)}\eta)(X_1 \dots X_{p+1}) =& \sum_i (-1)^{i+1} [\![\mathrm{hor}(X_i), \eta(X_1 \dots \hat{X}_i \dots X_{p+1})]\!]_{\mathrm{S}} \\
&+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1 \dots \hat{X}_i \dots \hat{X}_j \dots X_{p+1}) \\
(\mathrm{d}^{(2,-1)}\eta)(X_1 \dots X_{p+2}) =& \sum_{\sigma \in \mathrm{Sh}(2,p)} (-1)^{|\sigma|+p} [\![\mathbb{F}(X^{\mathrm{hor}}_{\sigma(1)}, X^{\mathrm{hor}}_{\sigma(2)}), \eta(X_{\sigma(3)} \dots X_{\sigma(p+2)})]\!]_{\mathrm{S}}
\end{aligned}$$

for all $\eta \in \Omega^p(Q, \Gamma(\wedge^q \ker T\pi))$, where we used the Schouten bracket, and $\mathrm{Sh}(2,p)$ denotes the set of $(2,p)$-shuffles on $\{1, \dots p+2\}$. For a proof in the Poisson case see [CF10, Proposition 5.3], and in the Dirac case [Mat13, Proposition 4.2.8].

By Theorem 2.3.1 (a) and (b) we obtain that the zeroth-page of the Serre spectral sequence associated to the Lie subalgebroid $(\ker T\pi)^*$ is given by

$$(E_0^{p,q}, \mathrm{d}_0) \simeq (\Omega^p(Q, \Gamma(\wedge^q \ker T\pi)), \mathrm{d}^{(0,1)} = (-1)^p [\![\mathcal{W}, \cdot\,]\!]_{\mathrm{S}}).$$

Inspired by [Vor05], we will denote the cohomology of the Lie algebroid $(\ker T\pi)^*$ by

$$\mathrm{H}^\bullet_V(M, \mathcal{W}) := \mathrm{H}^\bullet((\ker T\pi)^*).$$

The subscript is suggestive for the fact that this is the cohomology of the subcomplex of the Lichnerowicz complex of $\mathcal{W}$ consisting of 'vertical' multivector fields

$$(\Gamma(\wedge^\bullet \ker T\pi), [\![\mathcal{W}, \cdot]\!]_{\mathrm{S}}) \subseteq (\Gamma(\wedge^\bullet TM), [\![\mathcal{W}, \cdot]\!]_{\mathrm{S}}).$$

By the heuristic interpretation from Remark 2.3.2, one can think about elements $c \in \mathrm{H}^\bullet_V(M, \mathcal{W})$ as smooth families $\{c_x\}_{x\in Q}$ of Poisson cohomology classes $c_x \in \mathrm{H}^\bullet(M_x, W_x)$. In degree 0, we obtain the space of Casimir functions of the Poisson structure $\mathcal{W}$

$$\mathrm{H}^0_V(M, \mathcal{W}) = \mathrm{H}^0(M, \mathcal{W}) = \mathrm{Cas}(M, \mathcal{W}) \subseteq \mathscr{C}^\infty(M).$$

By Theorem 2.3.1 (c)-(e), the $\mathscr{C}^\infty(Q)$-modules $\mathrm{H}^\bullet_V(M, \mathcal{W})$ have a flat TQ-connection ∇^H,

$$\nabla^H_X[V] := [\mathscr{L}_{X^{\mathrm{hor}}}(V)], \quad [V] \in \mathrm{H}^\bullet_V(M, \mathcal{W}),$$

and we have that the first page of the Serre spectral sequence is given by

$$(E_1^{p,q}, \mathrm{d}_1) \simeq (\Omega^p(Q, \mathrm{H}^q_V(M, \mathcal{W})), \mathrm{d}^{\nabla^H}). \tag{2.45}$$

Of course, d^{∇^H} is induced by $\mathrm{d}^{(1,0)}$. The differential for $q = 0$,

$$\mathrm{d}^{\nabla^H} : \Omega^p(Q, \mathrm{Cas}(M, \mathcal{W})) \to \Omega^{p+1}(Q, \mathrm{Cas}(M, \mathcal{W})),$$

was used intensively in [Vor01, Vor05].

For horizontally nondegenerate Poisson structures, a description of Poisson cohomology in degree one was obtained in [VBV18, Theorem 4.8]. We recover and extend this result to the Dirac setting in the following theorem, which is a direct consequence of Theorem 2.3.1.

Theorem 2.3.19 *Let D be a horizontally nondegenerate Dirac structure on the total space of a surjective submersion $\pi\colon M \to Q$, with coupling data $(\mathcal{W}, H, \mathbb{F})$. The second page of the Serre spectral sequence associated to the extension* (2.44) *is given by*

$$E_2^{(p,q)} \simeq \mathrm{H}^p(Q, \mathrm{H}^q_V(M, \mathcal{W})),$$

where the right-hand side is the cohomology of the complex (2.45).

Hence, Dirac cohomology in degree zero is given by ∇^H-flat $\mathcal{W}$-Casimir functions on M,

$$\mathrm{H}^0(D) \simeq E_2^{(0,0)} = \{f \in \mathrm{Cas}(M, \mathcal{W}) : \nabla^H f = 0\},$$

and in degree one, we have that

$$\mathrm{H}^1(D) \simeq E_3^{(0,1)} \oplus E_2^{(1,0)} = \ker\big(\mathrm{d}_2\colon E_2^{(0,1)} \to E_2^{(2,0)}\big) \oplus E_2^{(1,0)},$$

where

$$\begin{aligned} E_2^{(1,0)} &\simeq \mathrm{H}^1(Q, \mathrm{Cas}(M, \mathcal{W})), \\ E_2^{(0,1)} &\simeq \mathrm{H}^0(Q, \mathrm{H}^1_V(M, \mathcal{W})), \text{ and} \\ E_2^{(2,0)} &\simeq \mathrm{H}^2(Q, \mathrm{Cas}(M, \mathcal{W})). \end{aligned}$$

The differential d_2 which appears in the theorem is recovered by the general construction of the spectral sequence. Namely, consider a class $c \in \mathrm{H}^0(Q, \mathrm{H}^1_V(M, \mathcal{W}))$. Choose a representative $V \in \Gamma(\ker T\pi)$. Then $[\![\mathcal{W}, V]\!]_{\mathrm{S}} = 0$ and

$$\mathrm{d}^{\nabla^H}[V] = [\mathrm{d}^{(1,0)}V] = 0 \in \Omega^1(Q, \mathrm{H}^1_V(M, \mathcal{W})).$$

Hence, by Lemma 2.5.1 (d), there is $\theta \in \Omega^1(Q, \mathscr{C}^\infty(M))$ such that

$$\mathrm{d}^{(1,0)}V = -\mathrm{d}^{(0,1)}\theta = [\![\mathcal{W}, \theta]\!]_{\mathrm{S}}.$$

Then d_2 is given by

$$\mathrm{d}_2 c = [\mathrm{d}^{(2,-1)}V + \mathrm{d}^{(1,0)}\theta] = [-\mathscr{L}_V \mathbb{F} + \mathrm{d}^{(1,0)}\theta] \in \mathrm{H}^2(Q, \mathrm{Cas}(M, \mathcal{W})).$$

Finally, we also mention the following direct consequence of Theorem 2.3.15.

Corollary 2.3.20 *In the setting of Theorem 2.3.19, if the Ehresmann connection H is complete, then the assignment*

$$U \mapsto \mathcal{S}^q_{\mathcal{W}}(U) := \left\{[V] \in \mathrm{H}^q_V(\pi^{-1}(U), \mathcal{W}) : \nabla^H[V] = 0\right\}$$

is a locally constant sheaf on Q, and

$$E_2^{(p,q)} \simeq \mathrm{H}^p(Q, \mathcal{S}^q_{\mathcal{W}}).$$

Normal forms around presymplectic leaves

In this subsection, we briefly discuss the linearisation problem around leaves for Poisson and Dirac structures, and how this leads to horizontally nondegenerate structures. We then apply the Serre spectral sequence to calculate the cohomology in low degrees of a class of structures that are partially linearisable, and obtain infinitesimal versions of results in [Bra04, Vor05].

Horizontally nondegenerate Poisson and Dirac structures arise naturally in the study of normal forms for such structures around leaves [Vor01, Vor05, Wad08]. Namely, let D be a Dirac structure on a manifold M and (Q, ω_Q) be an embedded presymplectic leaf. Denote by $\pi\colon E := \nu_Q(M) \to Q$ the normal bundle of Q and consider a tubular neighbourhood $\iota\colon E \hookrightarrow M$ of Q. Then ι^*D is a Dirac structure on E which is horizontally nondegenerate around Q—because it is so along Q—so, by shrinking ι, we may assume that we have a horizontally nondegenerate Dirac structure on E, which, for simplicity, we denote again by D. Since the zero section is a presymplectic leaf, the associated coupling data $(\mathcal{W}, H, \mathbb{F})$ satisfies the following properties.

1. For each $x \in Q$, the Poisson structure $\mathcal{W}_x$ on $E_x = \pi^{-1}(x)$ vanishes at x, and coincides with the transverse Poisson structure to the leaf, constructed in [Wei83] in the Poisson setting and in [DW08] in the Dirac setting;
2. H is tangent to Q, i.e. $H\big|_Q = TQ$;
3. $\mathbb{F}\big|_Q = -\omega_Q$.

The vector bundle structure on $\pi\colon E \to Q$ can be used to build the **linearisation** of D. This was described in the Poisson case in [Vor01, Vor05]; here we follow [Mat13, Chapter 4] and [CM12]. We construct a path of horizontally non-degenerate Dirac structures on E

$$D_t := t \cdot \mu_t^*(e^{(t-1)\pi^*\omega_Q} D), \qquad t \in (0,1],$$

where $e^{(t-1)\pi^*\omega_Q} D$ denotes the gauge transform of D via the closed 2-form $(t-1)\pi^*\omega_Q \in \Omega^2(E)$, μ_t^* is the pullback of Dirac structures via the fibrewise multiplication $\mu_t\colon E \to E$, and finally we use the "rescaling" of Dirac structures, defined as $t \cdot (X + \xi) = t \cdot X + \xi$. Clearly $D_1 = D$. Each Dirac structure D_t is horizontally nondegenerate, with corresponding coupling data

$$\mathcal{W}_t := t \cdot \mu_t^*(\mathcal{W}), \qquad H_t := \mu_t^*(H), \qquad \mathbb{F}_t := \frac{1}{t}\big(\mu_t^*(\mathbb{F}) + (t-1)\pi^*\omega_Q\big).$$

Using these formulas, one easily shows that $D_0 := \lim_{t\to 0} D_t$ exists and is a horizontally nondegenerate Dirac structure. The corresponding coupling data, denoted by

$$\big(\mathcal{W}_{\text{lin}}, H_{\text{lin}}, \mathbb{F}_{\text{lin}} - \pi^*\omega_Q\big),$$

satisfies the following:

1. $\mathcal{W}_{\text{lin}}$ is a family of linear Poisson structures on the fibres of E, i.e. for each $x \in Q$, $(\mathcal{W}_{\text{lin}})_x$ is a linear Poisson structure on the vector space E_x, or equivalently, a Lie algebra structure on E_x^*, which is precisely the isotropy Lie algebra at x of the original Dirac structure D;

2. H_{lin} is a linear connection on E, hence, in particular, it is complete;

3. $\mathbb{F}_{\text{lin}}$ is a linear horizontal 2-form, i.e. it can be viewed as an element in $\Omega^2(Q, E^*)$.

The Dirac structure D_0 plays the role of the **first order approximation** of D around Q. It can be described more intrinsically using the Lie algebroid $D\big|_Q \Rightarrow Q$ (see [Vor01, Section 5.2] and [Mat13, Proposition 4.2.25]). Let us note that, even if we start with a Poisson structure, i.e. $D = \text{graph}(w)$, the first order approximation D_0 will correspond to a Poisson structure only on some neighbourhood U of Q, i.e. $D_0\big|_U = \text{graph}(w_{\text{lin}})$.

The **linearisation problem** asks whether the Dirac structures D and D_0 are isomorphic around Q. For Poisson structures, the isomorphisms are diffeomorphism which fix Q pointwise; in general, one should also allow for exact gauge transformations. For points, i.e. $Q = \{x\}$, this is the linearisation problem for Poisson structures around zeroes initiated in [Wei83]. In this setting, Conn's famous theorem [Con85] says that a Poisson structure w is linearisable around a zero $x \in M$, provided the isotropy Lie algebra T_x^*M is semisimple and compact. For symplectic leaves of Poisson structures, the linearisation problem was studied [Vor01, Bra04, Vor05, CM12]. The first approaches were based on Conn's theorem, and produced the following partial linearisation result (see [Bra04, Corollary 2.5] or [Vor05, Theorem 4.12]).

Theorem 2.3.21 *Let* (M, w) *be a Poisson manifold, and* (Q, ω_Q) *be an embedded symplectic leaf such that the isotropy Lie algebra at points* $x \in Q$ *is semisimple and compact. Then there is a tubular neighbourhood* $\iota\colon E \hookrightarrow M$ *of* Q *in* M *such that* $\iota^* w$ *is horizontally nondegenerate around* Q *with coupling data*

$$(\mathcal{W}_{\text{lin}}, H_{\text{lin}}, \mathbb{F}),$$

where $\mathcal{W}_{\text{lin}}$ *is fibrewise linear and* H_{lin} *is a linear connection on* E*.*

We apply the techniques of this section to deduce an infinitesimal version of this partial linearisation result. Namely, Theorem 2.3.21 shows that Poisson structures as in the statement can be deformed only in the direction of the $\mathbb{F}$-component. On the other hand, deformations of Poisson structures are infinitesimally encoded by the second Poisson cohomology group. A consequence of Theorem 2.3.24 below is that the second Poisson cohomology of a Poisson structure (E, w) as in Theorem 2.3.21 satisfies

$$\mathrm{H}^2(E, w) \simeq \mathrm{H}^2(Q, \mathrm{Cas}(E, \mathcal{W}_{\text{lin}})).$$

For this, we first recall some classical results. Let $\mathfrak{g}$ be a compact semisimple Lie algebra and denote the linear Poisson structure on its dual by $(\mathfrak{g}^*, w_{\text{lin}})$. The Poisson cohomology of these linear Poisson structures has been computed in [GW92, Theorem 3.2], and is given by

$$\mathrm{H}^\bullet(\mathfrak{g}^*, w_{\text{lin}}) \simeq \mathrm{Cas}(\mathfrak{g}^*, w_{\text{lin}}) \otimes \mathrm{H}^\bullet(\mathfrak{g}). \tag{2.46}$$

Consider a vector bundle $\pi\colon E \to Q$ endowed with a vertical, fiberwise linear Poisson structure $\mathcal{W}_{\text{lin}}$. Then the dual bundle E^* is a smooth bundle of Lie algebras. We assume that the bundle is locally trivial with typical fibre a Lie algebra denoted by $\mathfrak{g}$. Then Lie algebra cohomology of the fibres yields a vector bundle over Q with typical fibre $\mathrm{H}^\bullet(\mathfrak{g})$, denoted by

$$\mathcal{H}^\bullet(E^*) \to Q, \quad \mathcal{H}^\bullet(E^*)_x := \mathrm{H}^\bullet(E^*_x) \simeq \mathrm{H}^\bullet(\mathfrak{g}).$$

We have the following extension of [GW92, Theorem 3.2].

Lemma 2.3.22 *If* $\mathfrak{g}$ *is compact and semisimple, then we have an isomorphism*

$$\mathrm{H}^\bullet_{\mathrm{V}}(E, \mathcal{W}_{\text{lin}}) \simeq \mathrm{Cas}(E, \mathcal{W}_{\text{lin}}) \otimes_{\mathscr{C}^\infty(Q)} \Gamma(\mathcal{H}^\bullet(E^*)). \tag{2.47}$$

Proof. For the proof, one uses the homotopy operators constructed in [GW92] for the isomorphism (2.46) in a local trivialisation. These, when applied to smooth families of forms, yield smooth families of forms—see the Remark following the proof of [GW92, Lemma 3.6]. □

Next, we show that these bundles have canonical connections.

Lemma 2.3.23 *If* $\mathfrak{g}$ *is compact and semisimple, then the* $\mathscr{C}^\infty(Q)$*-module* $\mathrm{H}^\bullet_{\mathrm{V}}(E, \mathcal{W}_{\text{lin}})$ *has a canonical flat* TQ*-connection*

$$\nabla^{\text{can}}\colon \Gamma(TQ) \times \mathrm{H}^\bullet_{\mathrm{V}}(E, \mathcal{W}_{\text{lin}}) \to \mathrm{H}^\bullet_{\mathrm{V}}(E, \mathcal{W}_{\text{lin}}).$$

Moreover, ∇^{can} *preserves the submodules* $\mathrm{Cas}(E, \mathcal{W}_{\text{lin}})$ *and* $\Gamma(\mathcal{H}^\bullet(E^*))$*, and acts as a derivation with respect to the decomposition* (2.47)*, i.e. it satisfies*

$$\nabla^{\text{can}}_X(f \otimes c) = \nabla^{\text{can}}_X(f) \otimes c + f \otimes \nabla^{\text{can}}_X(c).$$

Proof. Consider any linear Ehresmann connection ∇ which preserves $\mathcal{W}_{\mathrm{lin}}$ in the sense that (2.43) holds for the corresponding linear Ehresmann connection H_{lin}. Then we obtain a map

$$\nabla \colon \Gamma(TQ) \times \Gamma(\wedge^\bullet \ker T\pi) \to \Gamma(\wedge^\bullet \ker T\pi), \qquad \nabla_X V := [X^{\mathrm{hor}}, V],$$

where $\cdot^{\mathrm{hor}}$ denotes the horizontal lift induced by ∇. Since X^{hor} is a Poisson vector field for $\mathcal{W}_{\mathrm{lin}}$, we have an induced map ∇^{can}_X in cohomology, which clearly preserves the submodules from the statement and satisfies the derivation rule.

We show that the connection ∇^{can}_X on cohomology is independent of choices. Let ∇' be a second connection preserving $\mathcal{W}_{\mathrm{lin}}$. Then, for each $X \in \Gamma(TQ)$, we have a vertical Poisson vector field

$$\theta(X) := [X^{\mathrm{hor}} - X^{\mathrm{hor}'}] \in \mathrm{H}^1_{\mathrm{V}}(E, \mathcal{W}_{\mathrm{lin}}) \simeq \mathrm{Cas}(E, \mathcal{W}_{\mathrm{lin}}) \otimes_{\mathscr{C}^\infty(Q)} \Gamma(\mathcal{H}^1(E^*)).$$

However, since the typical fibre $\mathfrak{g}$ is semisimple, the Whitehead Lemma gives that $\mathcal{H}^1(E^*) = 0$. Hence $\theta(X) = 0$, and so ∇_X and ∇'_X have the same action on $\mathrm{H}^\bullet_{\mathrm{V}}(E, \mathcal{W}_{\mathrm{lin}})$.

Flatness can be checked locally, where one can use the trivial connection ∇ corresponding to a trivialisation of the bundle of Lie algebras. □

By using Theorems 2.3.1 and 2.3.15, and Whitehead's Lemma, we obtain the following.

Theorem 2.3.24 *Let D be a horizontally nondegenerate Dirac structure on a vector bundle $\pi \colon E \to Q$ whose coupling data has the first two components linear*

$$\big(\mathcal{W}_{\mathrm{lin}}, H_{\mathrm{lin}}, \mathbb{F}\big).$$

Assume that the typical fibre of the bundle of Lie algebras E^* corresponding to $\mathcal{W}_{\mathrm{lin}}$ is semisimple and compact. Then the second page of the Serre spectral sequence associated to the extension (2.44) *is given by*

$$E_2^{(p,q)} \simeq \mathrm{H}^p(Q, \mathcal{S}^q_{\mathcal{W}_{\mathrm{lin}}}),$$

where $U \mapsto \mathcal{S}^q_{\mathcal{W}_{\mathrm{lin}}}(U)$ is the locally constant sheaf

$$\mathcal{S}^q_{\mathcal{W}_{\mathrm{lin}}}(U) := \big\{ f \otimes c \in \mathrm{Cas}(\pi^{-1}(U), \mathcal{W}_{\mathrm{lin}}) \otimes_{\mathscr{C}^\infty(U)} \Gamma(\mathcal{H}^q(E^*|_U)) \colon \nabla^{\mathrm{can}}(f \otimes c) = 0 \big\}.$$

In particular, for $i = 0, 1, 2$, we have that

$$\mathrm{H}^i(D) \simeq E_2^{(i,0)} \simeq \mathrm{H}^i(Q, \mathcal{S}^0_{\mathcal{W}_{\mathrm{lin}}}) \simeq \mathrm{H}^i(Q, \mathrm{Cas}(E, \mathcal{W}_{\mathrm{lin}})).$$

For Poisson structures, versions of this result have appeared in the literature. For instance, in degree $i = 1$, the theorem implies [VBV18, Claim 1.2]. As mentioned above, for $i = 2$, Poisson cohomology encodes infinitesimal deformations and the result is an infinitesimal version of [Bra04, Corollary 2.5] or [Vor05, Theorem 4.12] (stated above as Theorem 2.3.21). Moreover, [Vor05, Section 4] shows that a relative version of the group $\mathrm{H}^2(Q, \mathrm{Cas}(E, \mathcal{W}_{\mathrm{lin}}))$ encodes the obstructions for the linearisation problem.

2.4 Abelian extensions

A special case of (2.23) where the differential on page E_2 can be described more explicitly is that of an **abelian extension**, i.e. a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, Rightarrow] & A \arrow[r, "\Pi"] \arrow[d, Rightarrow] & B \arrow[r] \arrow[d, Rightarrow] & 0 \\ & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\pi"] & Q & \end{tikzcd} \tag{2.48}$$

of Lie algebroids in which $L \Rightarrow M$ has zero Lie bracket. These have been studied in [Mac05] in the case when $M = Q$ and $\pi = \mathrm{id}_M$. By using generalised representations, we extend the results obtained there to the case when A and B are over different bases.

Consider a representation $V \to M$ of A on which L acts trivially. Then, since L is abelian, we have that $\mathrm{H}^q(L, V) = \Omega^q(L, V)$, and so Theorem 2.3.1 yields

$$E_2^{p,q} \simeq \mathrm{H}^p(B, \Omega^q(L, V)),$$

where, as seen in Theorem 2.3.1, $\Omega^q(L, V)$ is a generalised B-representation.

Next, note that there is a generalised representation of B also on the $\mathscr{C}^\infty(Q)$-module $\Gamma(L)$ given by $\nabla_b^L = [a, \cdot]$, where $a \in \Gamma(A)$ is any lift of $b \in \Gamma(B)$, i.e. satisfies $\Pi \circ a = b \circ \pi$. In the case when $\pi = \mathrm{id}_M$ this is a classical representation; see e.g. [Mac05, Proposition 3.3.20].

To identify the differential on the second page, fix a **Lie algebroid Ehresmann connection** for the extension (2.48), i.e. a splitting $\sigma \colon M \times_Q B \to A$ of Π. We obtain a $\mathscr{C}^\infty(Q)$-linear map at the level of sections, also denoted $\sigma \colon \Gamma(B) \to \Gamma(A)$, such that $\sigma(b)$ is a lift of $b \in \Gamma(B)$. The **curvature** of σ is defined, for $b_1, b_2 \in \Gamma(B)$, as

$$\gamma(b_1, b_2) = [\sigma(b_1), \sigma(b_2)]_A - \sigma([b_1, b_2]_B) \in \Gamma(L). \tag{2.49}$$

The curvature is $\mathscr{C}^\infty(Q)$-bilinear, so $\gamma \in \Omega^2(B, \Gamma(L))$. The Jacobi identity implies that γ is closed for the generalised representation of B. The **extension class** of (2.48) is defined by

$$[\gamma] \in \mathrm{H}^2(B, \Gamma(L)).$$

Any other splitting is of the form $\sigma' = \sigma + \lambda$, with $\lambda \in \Omega^1(B, \Gamma(L))$, and has corresponding curvature $\gamma' = \gamma + \mathrm{d}_B\lambda$. Hence, the extension class is independent of the Lie algebroid Ehresmann connection, and $[\gamma] = 0$ if and only if the extension admits a flat Lie algebroid Ehresmann connection, in which case $A \simeq \tilde{B} \ltimes L$, where $\tilde{B} := \operatorname{Im} \sigma$.

The extension class determines the differential on the second page of the spectral sequence. More precisely, contraction with the closed 2-form $\gamma \in \Omega^2(B, \Gamma(L))$ induces a cochain map

$$\mathrm{i}_\gamma \colon \big(\Omega^\bullet(B, \Omega^q(L, V)), \mathrm{d}_B\big) \to \big(\Omega^{\bullet+2}(B, \Omega^{q-1}(L, V)), \mathrm{d}_B\big),$$

and therefore a map in cohomology, which depends only on $[\gamma]$. We have the following result, which generalises [Mac05, Theorem 7.4.11] (for $B = TQ$), [HS53, Theorem 8] (for Lie algebras), and [MZ22, Corollary 4.3] (for the anchor of a regular Lie algebroid).

Theorem 2.4.1 *Let an abelian extension $L \to A \to B$ as in* (2.48) *be given, and let V be a representation of A on which L acts trivially. Under the isomorphism of Theorem 2.3.1, the differential on the second page of the Serre spectral sequence,* $\mathrm{d}_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}$, *becomes*

$$(-1)^p \mathrm{i}_{[\gamma]} \colon \mathrm{H}^p(B, \Omega^q(L, V)) \to \mathrm{H}^{p+2}(B, \Omega^{q-1}(L, V)). \tag{2.50}$$

The proof is standard (see e.g. the proof of [MZ22, Proposition 4.2]) and so we omit it.

For use in Chapter 3, we note the following.

Corollary 2.4.2 *If the abelian extension $L \to A \to B$ splits as Lie algebroids, i.e. the extension class is zero, and $L = M \times \mathbb{R}$ is trivial with trivial representation of B, the cohomology of A is given by*

$$\mathrm{H}^\bullet(A) = \mathrm{H}^\bullet(B) \oplus \mathrm{H}^{\bullet-1}(B). \tag{2.51}$$

2.4.1 Vaisman's spectral sequence for regular Poisson manifolds

As an application, we discuss the spectral sequence associated to the bundle of isotropy Lie algebras on a regular Poisson manifold, introduced in [Vai90]. Recall that a Poisson manifold (M, w) carries a **singular symplectic foliation**, i.e. a decomposition into immersed, connected submanifolds endowed with symplectic structures,

$$M = \sqcup_S (S, \omega_S),$$

whose members are called **symplectic leaves**. The singular symplectic foliation is determined by the following conditions (see e.g. [CFM21, Proposition 1.8 and Theorem 4.1])

$$T_x S = \mathrm{im}\,(\sharp_x \colon T_x^* M \to T_x M), \qquad \omega_S(\sharp\alpha, \sharp\beta) = -w(\alpha, \beta), \qquad \text{for all } x \in S.$$

A **regular** Poisson manifold is a Poisson manifold (M, w) for which the bivector field w has constant rank. In this case, the symplectic leaves form a smooth (regular) foliation $\mathcal{F}$, endowed with a leafwise symplectic form $\omega \in \Omega^2(\mathcal{F})$. The pair $(\mathcal{F}, \omega)$ is called a **symplectic foliation**. This gives a one-to-one correspondence between regular Poisson manifolds and symplectic foliations.

For a regular Poisson structure, we have that $\ker \sharp = \nu_{\mathcal{F}}^*$, and this Lie subalgebroid has trivial bracket. So we have an abelian extension

$$0 \longrightarrow \nu_{\mathcal{F}}^* \longrightarrow T^* M \xrightarrow{\sharp} T\mathcal{F} \longrightarrow 0, \tag{2.52}$$

where we have used the notation from Example 2.2.9, as we will also do in the rest of the section. The Serre spectral sequence for Poisson cohomology of this extension was considered in this generality first in [Vai90]. The corresponding filtration is given by

$$\mathcal{F}^p_{\nu^*_{\mathcal{F}}}\Gamma(\wedge^{p+q}TM) = \Gamma(\wedge^p T\mathcal{F}) \wedge \Gamma(\wedge^q TM).$$

The following determines the differential on the second page.

Lemma 2.4.3 *Let (M, w) be a regular Poisson manifold with underlying foliation $\mathcal{F}$ and leafwise symplectic structure $\omega \in \Omega^2(\mathcal{F})$. The extension class corresponding to* (2.52) *is given by*

$$\mathrm{var}(\omega) := \mathrm{d}_1[\omega] \in \mathrm{H}^2(\mathcal{F}, \nu^*_{\mathcal{F}}),$$

where $\mathrm{d}_1 \colon \mathrm{H}^2(\mathcal{F}) \to \mathrm{H}^2(\mathcal{F}, \nu^*_{\mathcal{F}})$ is the differential on the first page of the spectral sequence corresponding to $T\mathcal{F} \subseteq TM$ from Example 2.2.9. Explicitly, if $\tilde{\omega} \in \Omega^2(M)$ is a 2-form extending ω, the curvature form corresponding to the splitting $\sigma := \tilde{\omega}^\flat \colon T\mathcal{F} \to T^*M$ is given by

$$\gamma \in \Omega^2(\mathcal{F}, \nu^*_{\mathcal{F}}), \qquad \gamma(X_1, X_2) := (\mathrm{d}\tilde{\omega})(X_1, X_2, \cdot\,) \in \nu^*_{\mathcal{F}}.$$

Proof. The definition of $\sharp$ and ω imply that $\sharp \mathrm{i}_X \tilde{\omega} = X$ for $X \in \Gamma(T\mathcal{F})$, i.e. $\sigma = \tilde{\omega}^\flat$ is a splitting of (2.52). Using this and standard formulas of Cartan calculus, we obtain the claimed result

$$\begin{aligned}\gamma(X_1, X_2) &= [\mathrm{i}_{X_1}\tilde{\omega}, \mathrm{i}_{X_2}\tilde{\omega}] - \mathrm{i}_{[X_1,X_2]}\tilde{\omega} \\ &= \mathscr{L}_{X_1}\mathrm{i}_{X_2}\tilde{\omega} - \mathrm{i}_{X_2}\mathrm{d}\mathrm{i}_{X_1}\tilde{\omega} - \mathrm{i}_{[X_1,X_2]}\tilde{\omega} \\ &= \mathrm{i}_{X_2}\mathrm{i}_{X_1}(\mathrm{d}\tilde{\omega}).\end{aligned}$$

□

Theorem 2.4.1 yields the following result.

Theorem 2.4.4 *Let (M, w) be a regular Poisson manifold with underlying foliation $\mathcal{F}$ and leafwise symplectic structure $\omega \in \Omega^2(\mathcal{F})$. The second page of the Serre spectral sequence corresponding to the extension* (2.52) *is given by*

$$\big(E_2^{p,q}, \mathrm{d}_2\big) \simeq \big(\mathrm{H}^p(\mathcal{F}, \wedge^q \nu_{\mathcal{F}}), (-1)^p \mathrm{i}_{\mathrm{var}(\omega)}\big).$$

In particular, for the first Poisson cohomology group, we have an isomorphism

$$\mathrm{H}^1(M, w) \simeq \mathrm{H}^1(\mathcal{F}) \oplus \ker\big(\mathrm{i}_{\mathrm{var}(\omega)} : \mathrm{H}^0(\nu_{\mathcal{F}}) \to \mathrm{H}^2(\mathcal{F})\big).$$

Remark 2.4.5 The spectral sequence for Poisson cohomology discussed here was used implicitly in [VK88, VK90] to calculate Poisson cohomology in low degrees around leaves which admit a product neighbourhood. In the generality of this section, the spectral sequence was introduced in [Vai90], where a version of Theorem 2.4.4 was obtained, but without the interpretation of the differential as the cup product with the extension class.

Remark 2.4.6 The extension class $\mathrm{var}(\omega) \in \mathrm{H}^2(\mathcal{F}, \nu_{\mathcal{F}}^*)$ plays an important role in the Poisson geometry of the symplectic foliation $(\mathcal{F}, \omega)$, see also [CFM19, Section 3.4] for an overview.

First, by pulling back the extension class to a symplectic leaf (S, ω_S), we obtain the class $\mathrm{i}_S^*(\mathrm{var}(\omega)) \in H^2(S, \nu_S^*)$, which can be interpreted as the **cohomological variation** of the symplectic forms on the leaves at S. For example, if the foliation was trivial around S, so that the symplectic foliation is isomorphic to the family $\{(S \times \{x\}, \omega_x)\}_{x \in U}$, where $U \subseteq \mathbb{R}^k$ is a neighbourhood of 0, then $i_S^*(\mathrm{var}(\omega)) \in H^2(S) \otimes \mathbb{R}^k$ has components

$$\big[\partial_{x_i}\omega_x\big|_{x=0}\big] \in H^2(S), \quad 1 \le i \le k.$$

In general, fix $x_0 \in S$ and let $\widetilde{S}_{x_0}$ be the universal cover of S, viewed as a principal $\pi_1(S, x_0)$-bundle. By pulling back $i_S^*(\mathrm{var}(\omega))$ to $\widetilde{S}$ and using the Bott connection to trivialise the normal bundle over $\widetilde{S}_{x_0}$, we obtain a $\pi_1(S, x_0)$-equivariant linear map

$$[\delta_S\omega] \colon \nu_{S,x_0} \longrightarrow H^2(\widetilde{S}_{x_0}). \tag{2.53}$$

In [CM13] this map is called the **cohomological variation** of the symplectic form at S and is shown to play an important role in the local structure of the symplectic foliation around S.

Next, by composing the isomorphim $\pi_2(S, x_0) \simeq \pi_2(\widetilde{S}_{x_0}, x_0)$ (coming from the long exact sequence in homotopy corresponding to the fibration $\widetilde{S}_{x_0} \to S$) with the Hurewicz map $\pi_2(\widetilde{S}_{x_0}, x_0) \to H_2(\widetilde{S}_{x_0}, \mathbb{Z})$, we see that the dual of (2.53) gives a group homomorphism

$$\partial_{x_0} \colon \pi_2(S, x_0) \longrightarrow \nu_{S,x_0}^*.$$

This map is the so-called **monodromy map** at x_0, which plays a crucial role in the integrability problem of the Poisson manifold by a symplectic groupoid; namely, integrability is equivalent to the image of all these maps being uniformly discrete in $\nu_{\mathcal{F}}^*$ [CF04]. The monodromy map has the geometric interpretation of being the **variation of symplectic area of spheres** at x_0, i.e.

$$\langle \partial_{x_0}[\sigma], v \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}^2} \sigma_t^*\omega\big|_{t=0}, \qquad [\sigma] \in \pi_2(S, x_0), \quad v \in \nu_{S,x_0}$$

where $\sigma_t \colon \mathbb{S}^2 \to M$ is a family of smooth leafwise spheres, with $[\sigma_0] = [\sigma]$ and sending the north pole $N \in \mathbb{S}^2$ to a curve $x_t = \sigma_t(N)$ with direction v, i.e. $[\frac{\mathrm{d}}{\mathrm{d}t}x_t\big|_{t=0}] = v$.

Remark 2.4.7 For a non-regular Poisson manifold (M, w), there are several different ways in which one can generalise the above filtration; at least the following.

First, consider the filtration by the powers of the ideal of Hamiltonian vector fields

$$\mathcal{F}_{\mathrm{ham}}^k := \wedge^k \mathcal{I}_{\mathrm{ham}}, \qquad \mathcal{I}_{\mathrm{ham}} := \big\{\sharp\mathrm{d}f \wedge \vartheta : f \in \mathscr{C}^\infty(M), \vartheta \in \Gamma(\wedge TM)\big\}.$$

It is easy to see that $\mathcal{I}_{\mathrm{ham}}$ can also be generated by vector fields of the form $\sharp\alpha$, with $\alpha \in \Gamma(T^*M)$. The corresponding spectral sequence is mentioned in [Vai90].

Secondly, by considering tangential vector fields, one obtains a potentially larger filtration

$$\mathcal{F}^k_{\mathrm{tang}} := \wedge^k \mathcal{I}_{\mathrm{tang}}, \qquad \mathcal{I}_{\mathrm{tang}} := \big\{X \wedge \vartheta : X \in \Gamma(T\mathcal{F}), \vartheta \in \Gamma(\wedge^\bullet TM)\big\},$$

where $\Gamma(T\mathcal{F})$ denotes the set of vector fields tangent to the symplectic leaves of w. The corresponding spectral sequence was used in the proof of [Gin96, Theorem 4.15]. In general, the inclusion $\mathcal{F}^k_{\mathrm{ham}} \subseteq \mathcal{F}^k_{\mathrm{tang}}$ is strict. Indeed, for the Poisson structure $w := (x^2 + y^2)\partial_x \wedge \partial_y$ on $\mathbb{R}^2$, we have that $x\partial_x \in \mathcal{I}_{\mathrm{tang}} \backslash \mathcal{I}_{\mathrm{ham}}$.

Next, the conormal bundle of any symplectic leaf S is a Lie subalgebroid $\iota_{\nu^*_S} : \nu^*_S \to T^*M$, and so it gives a differential graded ideal $\mathcal{I}_{\nu^*_S} := \ker(\iota^*_{\nu^*_S}) \subseteq \Gamma(\wedge^\bullet TM)$. We obtain a filtration

$$\mathcal{F}^k_{\nu^*} := \bigcap_S \wedge^k \mathcal{I}_{\nu^*_S},$$

where the intersection is over all leaves. In general, the inclusion $\mathcal{F}^k_{\mathrm{tang}} \subseteq \mathcal{F}^k_{\nu^*}$ is strict. For example, for the Poisson manifold $(\mathbb{R}^2, w = x\partial_x \wedge \partial_y)$, we have that $w \in \mathcal{F}^2_{\nu^*} \backslash \mathcal{F}^2_{\mathrm{tang}}$.

Next, consider multivector fields that satisfy pointwise the tangential condition

$$\mathcal{F}^k_{\mathrm{pt}} := \big\{\vartheta : \vartheta_x \in \wedge^k T_x\mathcal{F} \wedge \ \dots\ , \text{ for all } x \in M\big\}.$$

This filtration has appeared in [Vai94, Proposition 5.5] in the following equivalent form

$$\mathcal{F}^k_{\mathrm{pt}} := \big\{\vartheta : \vartheta(\alpha_1, \dots, \alpha_n) = 0, \text{ for all } \alpha_1, \dots \alpha_{n+1-k} \in \ker \sharp_x,\ x \in M\big\}.$$

We show that $\mathcal{F}^k_{\mathrm{pt}}$ is indeed a filtration by differential graded ideals. For this, we use the Lie subalgebroid $\iota_{T^*M|_S} : T^*M|_S \to T^*M$ corresponding to a symplectic leaf S and the differential graded ideal corresponding to the inclusion $j_{\nu^*_S} : \nu^*_S \hookrightarrow T^*M|_S$,

$$\mathcal{J}_{\nu^*_S} := \ker\big(j^*_{\nu^*_S} : \Gamma(\wedge^\bullet TM|_S) \to \Gamma(\wedge^\bullet \nu_S)\big).$$

Then we have the following description of the filtration

$$\mathcal{F}^k_{\mathrm{pt}} = \bigcap_S (\iota^*_{T^*M|_S})^{-1}(\wedge^k \mathcal{J}_{\nu^*_S}).$$

Clearly, $(\iota^*_{T^*M|_S})^{-1}(\mathcal{J}_{\nu^*_S}) = \mathcal{I}_{\nu^*_S}$, but for powers of the ideals, the analogue equality might fail if S is not an embedded leaf. Therefore we expect also the inclusion $\mathcal{F}_{\nu^*} \subseteq \mathcal{F}_{\mathrm{pt}}$ to be strict.

We can build another filtration by using the regular part of M, i.e. the open set M_{reg} consisting of points in M where the rank of w is locally constant. Namely, define

$$\mathcal{F}^k_{\mathrm{reg}} := \{\vartheta : \vartheta|_{M_{\mathrm{reg}}} \in \mathcal{F}^k(M_{\mathrm{reg}})\},$$

where $\mathcal{F}^k(M_{\text{reg}})$ denotes any of the above filtrations for the regular Poisson manifold M_{reg}. It is easy to see that the inclusion $\mathcal{F}^k_{\text{pt}} \subseteq \mathcal{F}^k_{\text{reg}}$ is in general strict.

2.5 Appendix

Lemma 2.5.1 *Let $V \to Q$ be a vector bundle. We have canonical isomorphisms:*

1. *For any vector bundle $W \to Q$,*

$$\Gamma(V) \otimes_{\mathscr{C}^\infty(Q)} \Gamma(W) \simeq \Gamma(V \otimes W);$$

2. *For any $\mathscr{C}^\infty(Q)$-module $\mathfrak{M}$,*

$$\Gamma(V) \otimes_{\mathscr{C}^\infty(Q)} \mathfrak{M} \simeq \mathrm{Hom}_{\mathscr{C}^\infty(Q)}\big(\Gamma(V^*), \mathfrak{M}\big);$$

3. *For any smooth map $\phi : M \to Q$,*

$$\Gamma(V) \otimes_{\mathscr{C}^\infty(Q)} \mathscr{C}^\infty(M) \simeq \Gamma(\phi^* V);$$

4. *For any cochain complex $(\mathcal{C}^\bullet, \mathrm{d})$ of $\mathscr{C}^\infty(Q)$-modules,*

$$\Gamma(V) \otimes_{\mathscr{C}^\infty(Q)} \mathrm{H}^\bullet(\mathcal{C}) \simeq \mathrm{H}^\bullet(\Gamma(V) \otimes_{\mathscr{C}^\infty(Q)} \mathcal{C}).$$

Proof. The statements hold because $\Gamma(V)$ is a finitely generated, projective $\mathscr{C}^\infty(Q)$-module. More precisely, the statements are obviously satisfied when V is trivial, and in general, they follow because we can write V as a subbundle of a trivial bundle $\mathbb{R}^n_Q \to Q$, with inclusion map $i\colon V \to \mathbb{R}^n_Q$ and projection map $p\colon \mathbb{R}^n_Q \to V$.

For the first claim, we need to show that the obvious map

$$\varphi\colon \Gamma(V) \otimes_{\mathscr{C}^\infty(Q)} \Gamma(W) \to \Gamma(V \otimes W)$$

is an isomorphism. Clearly, for $V = \mathbb{R}^n_Q$ the map φ is an isomorphism. Denote the inverse by

$$\psi\colon \Gamma(\mathbb{R}^n_Q \otimes W) \to \Gamma(\mathbb{R}^n_Q) \otimes_{\mathscr{C}^\infty(Q)} \Gamma(W).$$

Then the inverse of φ is given by

$$\varphi^{-1} = (p_* \otimes \mathrm{id}_{\Gamma(W)}) \circ \psi \circ (i \otimes \mathrm{id}_W)_*.$$

The same scheme can be used to prove the other isomorphisms. ☐

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Chapter 3

The blowdown map in cohomology

This chapter contains [Sch24], submitted, with the following changes. The introductory section [Sch24, Section 2.1] for Lie algebroids has been removed, we refer to Section 1.1.2 instead.

Moreover, we added Section 3.7, where we classify pullback functions and sections of real projective blowups.

Abstract

We study the blowdown map in cohomology in the context of real projective blowups of Lie algebroids. Using the blowdown map in cohomology we compute the Lie algebroid cohomology of the blowup of transversals of arbitrary codimension, generalising the Mazzeo-Melrose theorem on b-cohomology. To prove the result, we develop a Gysin sequence for Lie algebroids. As another example we use the developed tools to compute the Lie algebroid cohomology of the action Lie algebroid $\mathfrak{so}(3) \ltimes \mathbb{R}^3$, a result known in Poisson geometry literature. Moreover, we use similar techniques to compute the de Rham cohomology of real projective blowups.

3.1 Introduction

Blowup constructions are well-known in algebraic geometry (see e.g. [Har77]). The idea is to replace a point or subvariety by all lines normal to it. A famous result by Hironaka [Hir64a, Hir64b] states that in characteristic zero, one can always desingularise algebraic varieties, i.e. obtain a smooth variety, by a sequence of blowups. In the context of smooth manifolds, singularities often arise from additional geometric structures (a Poisson structure, a foliation, etc.).

There are various kinds of blowup constructions for smooth manifolds known in the literature. Spherical blowup, which replaces a submanifold by the sphere bundle of its normal bundle, has been used in Melrose's b-calculus (see [Mel93], in particular Chapter 4). Furthermore, it has been used to resolve singularities in the context of Lie groupoids [AM11, Nis19].

In this paper we focus on real projective blowups of smooth manifolds, in which the submanifold is replaced by the projectivisation of its normal bundle. Of particular interest are blowups of Lie algebroids, for which the construction is given in [DS21] ([GL13] for a base of codimension 1), see also [Obs21]. Given a Lie algebroid $A \Rightarrow M$ and a Lie subalgebroid $B \Rightarrow N$ over a submanifold $N \subseteq M$, blowing up yields a new Lie algebroid $\mathrm{Blup}(A, B)$ over the base $\mathrm{Blup}(M, N)$ together with a Lie algebroid morphism back to A, the blowdown map,

$$\begin{tikzcd}
\mathrm{Blup}(A,B) \arrow[r, "p_A"] \arrow[d, Rightarrow] & A \arrow[d, Rightarrow] \\
\mathrm{Blup}(M,N) \arrow[r, "p"] & M
\end{tikzcd}$$

Note that there exists a corresponding construction for Lie groupoids [DS21], see also [Obs21]. The projective blowup construction has been used to desingularise proper groupoids [PTW21, Wan18], and in the context of Lie algebroids blowing up has been shown to recover interesting Lie algebroids (a construction called *elementary modification* in [GL13, Definition 2.11]), like log- or scattering tangent bundles (see also [Kla17, Lan21]).

In this chapter we seek to gain insights into Lie algebroid cohomologies using real projective blowups. The blowdown map induces a map between the respective cohomology groups

$$p_A^* \colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(\mathrm{Blup}(A, B)). \tag{3.1}$$

It is this blowdown map in cohomology (3.1) we aim to study in this chapter: Since it relates two cohomology groups, understanding of the map allows to gain information about one cohomology group via the other. This works in two ways: We either start with a known Lie algebroid and are interested in the cohomology of the blowup, or it is the cohomology of the original Lie algebroid we are interested in. In this case, we choose the Lie subalgebroid in such a way that the cohomology of the blowup is easier to compute than that of A, e.g. such that $\mathrm{Blup}(A, B)$ is a regular Lie algebroid. Ultimately, we hope that the combination of desingularisation using blowups and understanding of the blowdown map in cohomology leads to a new way of computing Lie algebroid cohomologies.

In general, we can fit the blowdown map into the short exact sequence

$$0 \longrightarrow \Omega^\bullet(A) \xrightarrow{p_A^*} \Omega^\bullet(\mathrm{Blup}(A, B)) \longrightarrow \frac{\Omega^\bullet(\mathrm{Blup}(A, B))}{p_A^*\Omega^\bullet(A)} \longrightarrow 0$$

of cochain complexes. This short exact sequence induces a long exact sequence in cohomology, given by

$$\ldots \longrightarrow \mathrm{H}^\bullet(A) \xrightarrow{p_A^*} \mathrm{H}^\bullet(\mathrm{Blup}(A, B)) \xrightarrow{f} \mathrm{H}^\bullet\left(\frac{\Omega^\bullet(\mathrm{Blup}(A, B))}{p_A^*\Omega^\bullet(A)}\right) \longrightarrow \ldots \tag{3.2}$$

with the following property, see Theorem 3.4.1 for details.

Theorem 3.1.1 (The blowdown map in cohomology) *For the long exact sequence* (3.2) *there is a canonical isomorphism*

$$\mathrm{H}^\bullet\left(\frac{\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\Omega^\bullet(A)}\right) \simeq \mathrm{H}^\bullet\left(\frac{\mathscr{J}^\infty_{\mathbb{P}(\nu_N(M))}\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\mathscr{J}^\infty_N\Omega^\bullet(A)}\right). \tag{3.3}$$

Here, we write

$$\mathscr{J}^\infty_N\Omega^\bullet(A) = \Omega^\bullet(A)/\bigcap_{k\in\mathbb{N}}\mathcal{I}^k_N\Omega^\bullet(A) \tag{3.4}$$

for ∞-jets of forms of A along N, where $\mathcal{I}_N$ denotes the vanishing ideal of N.

Theorem 3.1.1 also implies the following. If $\iota\colon U \hookrightarrow \mathrm{Blup}(M,N)$ is an open neighbourhood of $\mathbb{P}(\nu_N(M))$, then $f = f_U \circ \mathrm{H}(\iota^*)$. Here,

$$\mathrm{H}(\iota^*)\colon \mathrm{H}^\bullet(\mathrm{Blup}(A,B)) \to \mathrm{H}^\bullet(\mathrm{Blup}(A,B)\big|_U)$$

denotes the map induced in cohomology by the inclusion and f_U is the map corresponding to f in the long exact sequence (3.2) for $A\big|_{p(U)}$, see again Theorem 3.4.1 for more details.

As a first application of Theorem 3.1.1 we express the cohomology of the blowup of a transversal in terms of $\mathrm{H}^\bullet(A)$. The proof uses that transversals admit a simple normal form [BLM16].

Theorem 3.1.2 (The blowup of transversals) *Let $\iota\colon N \hookrightarrow M$ be a closed transversal of $A \Rightarrow M$ and denote the projection of the projective bundle $\mathbb{P}(\nu_N(M)) \subseteq \mathrm{Blup}(M,N)$ by $\pi_\mathbb{P}\colon \mathbb{P}(\nu_N(M)) \to N$. Let $\iota^! A \Rightarrow N$ be the pullback of A to N.*

1. *If* codim N *is odd, we have an isomorphism*

   $$\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)) \simeq \mathrm{H}^\bullet(A) \oplus \mathrm{H}^{\bullet-1}(\iota^!A) \tag{3.5}$$

   and, under (3.5), *p_A^* becomes the isomorphism $p_A^*\colon \mathrm{H}^\bullet(A) \xrightarrow{\sim} \mathrm{H}^\bullet(A)\oplus 0$.*
2. *If* codim N *is even, there exists a tubular neighbourhood $E \to N$ such that* $\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A))$ *fits into a long exact sequence*

   $$\ldots \to \mathrm{H}^\bullet(A) \xrightarrow{p_A^*} \mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)) \to$$
   $$\to \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A\big|_E) \oplus \mathrm{H}^{\bullet-1}(\pi_\mathbb{P}^!A) \xrightarrow{g} \mathrm{H}^{\bullet+1}(A) \to \ldots$$

   where $g = i \circ \mathrm{pr}_{\mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A|_E)}$. Here, by $\mathrm{H}^\bullet_{\mathrm{cv}}(A\big|_E)$ we denote compact vertical cohomology and by $i\colon \mathrm{H}^\bullet_{\mathrm{cv}}(A\big|_E) \to \mathrm{H}^\bullet(A)$ the natural map.

Recall that by [GL13, Section 2.4.1] we can write the b-tangent bundle associated to a closed hypersurface $N \subseteq M$ by

$$T^b_N M = \mathrm{Blup}(TM, TN),$$

which is a blowup of a codimension 1 transversal. Theorem 3.1.2 then reproduces the Mazzeo-Melrose decomposition for b-cohomology [GMP14, MT14], see [Mel93] for the original result. In this sense, Theorem 3.1.2 can be seen as a generalisation of Mazzeo-Melrose as it allows for arbitrary Lie algebroids and transversals of arbitrary codimension.

One of the ingredients needed for proving the second part of Theorem 3.1.2 is the existence of a Gysin-like long exact sequence for the cohomology of the pullback of a Lie algebroid to a sphere bundle.

Theorem 3.1.3 (Gysin sequence for Lie algebroids) *Let $B \Rightarrow N$ be a Lie algebroid with anchor $\sharp$, $\pi \colon \mathbb{S} \to N$ a sphere bundle of rank k with orientation bundle $o(\mathbb{S}) \to N$, and $\pi^! B \Rightarrow \mathbb{S}$ the pullback Lie algebroid. There exists a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^\bullet(B) \xrightarrow{(\pi^!)^*} \mathrm{H}^\bullet(\pi^! B) \xrightarrow{(\pi^!)_*} \mathrm{H}^{\bullet-k}(B, o(\mathbb{S})) \xrightarrow{\wedge \sharp^* e} \mathrm{H}^{\bullet+1}(B) \longrightarrow \ldots \tag{3.6}$$

Here, $(\pi^!)_*$ denotes fibre integration and $e \in \mathrm{H}^{k+1}(N, o(\mathbb{S}))$ is the Euler class of the sphere bundle.

A result we obtain while proving Theorem 3.1.2 is on the de Rham cohomology of real projective blowups. For complex projective blowups there exist results on the de Rham cohomology, see e.g. [GH78], but for real projective blowups we could not find the statement of Theorem 3.1.4 in the literature.

Theorem 3.1.4 (de Rham cohomology of real projective blowups) *Let $N \subseteq M$ be a closed submanifold.*

1. *If $\operatorname{codim} N$ is odd, then we have an isomorphism*
$$p^* \colon \mathrm{H}^\bullet(M) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathrm{Blup}(M, N)). \tag{3.7}$$
2. *If $\operatorname{codim} N$ is even, let $E \to N$ be a tubular neighbourhood of N in M. Then $\mathrm{H}^\bullet(\mathrm{Blup}(M,N))$ fits into a long exact sequence*
$$\cdots \to \mathrm{H}^\bullet(M) \xrightarrow{p^*} \mathrm{H}^\bullet(\mathrm{Blup}(M,N)) \xrightarrow{h} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(E) \xrightarrow{i} \mathrm{H}^{\bullet+1}(M) \to \ldots \tag{3.8}$$
Here, h first restricts a form to $\mathbb{P}(\nu_N(M))$, then fibre-integrates and applies the Thom isomorphism.

On the other hand, in some cases blowing up orbits of Lie algebroids with singular orbit foliation leads to regular Lie algebroids, whose cohomology is easier to compute.

A particular example is the action Lie algebroid $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ with singular orbit given by the origin. We compute its cohomology by blowing up, reproducing the result on cohomology obtained by averaging in [GW92].

Theorem 3.1.5 (The action Lie algebroid $\mathfrak{so}(3) \ltimes \mathbb{R}^3$) *Let $A = \mathfrak{so}(3) \ltimes \mathbb{R}^3$ and denote by $A\big|_{\{0\}} \simeq \mathfrak{so}(3)$ its restriction to the origin.*

1. *The blowup $\mathrm{Blup}(A, A\big|_{\{0\}})$ is a regular Lie algebroid with cohomology given by*
$$\mathrm{H}^k(\mathrm{Blup}(A, A\big|_{\{0\}})) \simeq \begin{cases} \mathscr{A} & \text{if } k = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$
where $\mathscr{A} = \{f \in \mathscr{C}^\infty(\mathbb{R}^3) : f \text{ only depends on the radius}\}$

2. *The blowdown map in cohomology*

$$p_A^*\colon \mathrm{H}^\bullet(A) \xrightarrow{\simeq} \mathrm{H}^\bullet(\mathrm{Blup}(A, A|_{\{0\}}))$$

is an isomorphism.

Organisation of the chapter

In Section 3.4 we prove Theorem 3.1.1 while studying the blowdown map in cohomology (3.1). Next, we use Theorem 3.1.1 to compute the cohomology of the blowup of transversals in Section 3.5, including Theorem 3.1.2 and 3.1.4. In Section 3.6 we investigate two particular cases of blowups of invariant submanifolds. We prove Theorem 3.1.5 and conclude the section by showing that, by repeatedly blowing up restrictions of an action Lie algebroid to orbits one does not always obtain a regular Lie algebroid using $A = \mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^3$ in Section 3.6.2.

Finally, in the appendix we discuss fibre integration for Lie algebroids to be able to formulate and prove Theorem 3.1.3.

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3.2 Lie algebroids and pullbacks

In this section we prove an identification for the Lie algebroid cohomology of the pullback of a Lie algebroid to a double cover of its base in Lemma 3.2.1, which we use repeatedly throughout this paper. The material presented in this section is standard, see e.g. [Mac05] for the general theory of Lie algebroids.

Recall from Section 1.1.2 that, if $A \Rightarrow M$ is a Lie algebroid and $\phi\colon N \to M$ is a map transverse to the anchor, we can form the pullback Lie algebroid $\phi^! A \Rightarrow N$. Given a covering map ϕ, the transversality condition is trivially fulfilled. Since in this case, $T_{\tilde{x}}\phi\colon TN \to TM$ is a bijection, the TN part of a point in $\phi^! A$ is uniquely determined by the point in A, hence as a vector bundle $\phi^! A = \phi^\sharp A$ is just the pullback vector bundle.

Now, let $\phi\colon \tilde{M} \to M$ be a double cover and denote by $\{x^+, x^-\} \subseteq \tilde{M}$ the preimage of $x \in M$ under ϕ. Recall that for any vector bundle $V \to M$ the pullback bundle $\phi^\sharp V$ carries a canonical $\mathbb{Z}_2$-action, where the nontrivial element $\hat{1} \in \mathbb{Z}_2$ sends $v_{x^\pm}$ to $v_{x^\mp}$, i.e. exchanges the base point without changing the fibre. The induced $\mathbb{Z}_2$-action on $\Omega^\bullet(\phi^! A, \phi^\sharp V)$ is given by

$$(\hat{1}.\tilde{\omega})(\tilde{a}_1, \dots, \tilde{a}_k) = \hat{1}.\big(\tilde{\omega}(\hat{1}.\tilde{a}_1, \dots, \hat{1}.\tilde{a}_k)\big), \tag{3.9}$$

where $\tilde{\omega} \in \Omega^k(\phi^! A, \phi^\sharp V)$ and $\tilde{a}_1, \dots, \tilde{a}_k \in \Gamma(\phi^! A)$.

Lemma 3.2.1 *Let $A \Rightarrow M$ be a Lie algebroid with a representation on a vector bundle $V \to M$ and $\phi\colon \tilde{M} \to M$ a double cover.*

1. *The action of $\mathbb{Z}_2$ on $\Omega^\bullet(\phi^! A, \phi^\sharp V)$ is by chain maps. The $+1$-eigenspace of $\hat{1} \in \mathbb{Z}_2$ is given by*
$$\Omega^\bullet(\phi^! A, \phi^\sharp V)_+ = (\phi^!)^*\Omega^\bullet(A, V) \simeq \Omega^\bullet(A, V). \tag{3.10}$$

2. *Let $L = \tilde{M} \times_{\mathbb{Z}_2} \mathbb{R} \to M$, where $\hat{1}.(x^\pm, \lambda) = (x^\mp, -\lambda)$. Consider the representation of A on L defined by flatness of locally constant sections. Then the -1-eigenspace of $\hat{1} \in \mathbb{Z}_2$ is given by*
$$\Omega^\bullet(\phi^! A, \phi^\sharp V)_- \simeq \Omega^\bullet(A, V \otimes L). \tag{3.11}$$

Note that the condition on the representation of A on L in the second part of Lemma 3.2.1 is nontrivial as it implies that the representation of A on L factors through the anchor, i.e. there is a (vector bundle) connection ∇^{TM} on L such that
$$\nabla_a = \nabla^{TM}_{\sharp(a)} \tag{3.12}$$
for all $a \in \Gamma(A)$.

Proof of Lemma 3.2.1. To see that $\mathbb{Z}_2$ acts by chain maps is straightforward when evaluating a form on pullback sections. To show the identifications of complexes, consider first the $+1$ eigenspace. Clearly, pullback forms are invariant. On the other hand, if $\tilde{\omega} \in \Omega^k(\phi^! A, \phi^\sharp V)_+$ we can define $\omega \in \Omega^k(A, V)$ by
$$\omega(a_1, \dots, a_k)\colon M \ni x \mapsto \tilde{\omega}_{\tilde{x}}(\phi^\sharp a_1(\tilde{x}), \dots, \phi^\sharp a_k(\tilde{x})) \in \phi^\sharp V_{\tilde{x}} = V_x,$$
where $a_1, \dots, a_k \in \Gamma(A)$ and $\tilde{x} \in \phi^{-1}(\{x\})$. This is well-defined by $\mathbb{Z}_2$-invariance of $\tilde{\omega}$ and $(\phi^!)^*\omega = \tilde{\omega}$, proving the first part.

For the second part L be given as stated. The line bundle L is trivial if and only if $\tilde{M} \to M$ is the trivial double cover and the representation of A on L is trivial, implying
$$\Omega^\bullet(\phi^! A, \phi^\sharp V)_- \simeq \Omega^\bullet(\phi^! A, \phi^\sharp V)_+ \simeq \Omega^\bullet(A, V) \simeq \Omega^\bullet(A, V \otimes L)$$
using the first part. If L is nontrivial, then the pullback bundle $\phi^\sharp L$ is trivial. In a trivialisation $\phi^\sharp L = \tilde{M} \times \mathbb{R}$ for every $v_x \in L_x$ we have that
$$(\phi^\sharp)^{-1}(\{v_x\}) = \{(x^+, r(v_x)), (x^-, -r(v_x))\} \subseteq \tilde{M} \times \mathbb{R}, \tag{$*$}$$
where $\{x^+, x^-\} = \phi^{-1}(\{x\})$, and the $\mathbb{Z}_2$-action flips the two points. Let $\omega = \eta \otimes \ell \in \Omega^k(A, V \otimes L)$ be given, where $\eta \in \Omega^k(A, V)$ and $\ell \in \Gamma(L)$. Then we can define a form $\tilde{\omega} \in \Omega^k(\phi^! A, \phi^\sharp V)$ by
$$\begin{aligned}\tilde{\omega}(\tilde{a}_1, \dots, \tilde{a}_k)\colon \tilde{M} &\to V_{\phi(x^\pm)} = \phi^\sharp V_{x^\pm}\\ x^\pm &\mapsto \pm r\big(\ell(\phi(x^\pm))\big)\eta\big(\phi^!(\tilde{a}_1(x^\pm)), \dots, \phi^!(\tilde{a}_k(x^\pm))\big).\end{aligned}$$
where $\tilde{a}_1, \dots, \tilde{a}_k \in \Gamma(\phi^! A)$. $\mathscr{C}^\infty(M)$-linear extension as a module morphism along ϕ^* gives a map $\Omega^\bullet(A, V \otimes L) \to \Omega^\bullet(\phi^! A, \phi^\sharp V)$ which maps into the

subcomplex of anti-invariant forms. To show that it is a bijection, let $\tilde{\omega} \in \Omega^k(\phi^! A, \phi^\sharp V) = \Omega^k(\phi^! A, \phi^\sharp(V \otimes L))$ be given. For $a_1, \dots, a_k \in \Gamma(A)$ we set

$$\omega(a_1, \dots, a_k) \colon x \mapsto \phi^\sharp\big(\tilde{\omega}_{\tilde{x}}(\tilde{a}_1(\tilde{x}), \dots, \tilde{a}_k(\tilde{x}))\big), \tag{3.13}$$

where $\tilde{a}_j(\tilde{x})$ is chosen such that $\phi^!(\tilde{a}_j(\tilde{x})) = a_j(x)$. This map is well-defined by $\mathbb{Z}_2$-anti invariance of $\tilde{\omega}$ and defines a smooth section since ϕ is a local diffeomorphism. Clearly, the two constructions are inverses to each other. Finally, one needs to check that this construction gives a chain map, but this follows from the definition of the pullback representation. □

Finally, we note that the (anti-) invariant part of the cohomology is the cohomology of the (anti-) invariant subcomplex.

Lemma 3.2.2 *Let $A \Rightarrow M$, $\phi \colon \tilde{M} \to M$, $V \to M$ and $L \to M$ be given as in Lemma 3.2.1. Then*

$$\mathrm{H}^\bullet(\phi^! A, \phi^\sharp V)_\pm \simeq \mathrm{H}^\bullet(\Omega^\bullet(\phi^! A, \phi^\sharp V)_\pm). \tag{3.14}$$

Proof. This is obtained by averaging the $\mathbb{Z}_2$-action: For example, to see that the natural map for the invariant eigenspaces is injective suppose that $[\omega]_+ \in \mathrm{H}^\bullet(\Omega^\bullet(\phi^! A, \phi^\sharp V)_+)$. If $[\omega] = 0$, there exists $\theta \in \Omega^{\bullet-1}(\phi^! A, \phi^\sharp V)$ with $\mathrm{d}\theta = \omega$. But then

$$\tfrac{1}{2}(\theta + \hat{1}.\theta) \in \Omega^{\bullet-1}(\phi^! A, \phi^\sharp V)_+$$

is an invariant primitive for ω. □

3.3 Blowup of Lie algebroids

The idea of the blowup of a Lie algebroid is the following. Suppose that $B \Rightarrow N$ is a Lie subalgebroid of $A \Rightarrow M$ over a closed and embedded submanifold $N \subseteq M$. Then blowing up will lead to a new Lie algebroid $\mathrm{Blup}(A, B) \Rightarrow \mathrm{Blup}(M, N)$ together with a morphism of Lie algebroids $p_A \colon \mathrm{Blup}(A, B) \to A$. Hence, we get an induced map in cohomology $p_A^* \colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(\mathrm{Blup}(A, B))$. Now two scenarios are possible: Either the blown-up Lie algebroid is of interest and we would like to understand its cohomology in relation to $\mathrm{H}^\bullet(A)$, or we are interested in computing $\mathrm{H}^\bullet(A)$. In the second case, one chooses the Lie subalgebroid B in such a way that $\mathrm{H}^\bullet(\mathrm{Blup}(A, B))$ is easier to compute and then draws conclusions regarding $\mathrm{H}^\bullet(A)$ via the blow-down map.

3.3.1 Real projective blowups of smooth manifolds

Throughout this section let $N \subseteq M$ be closed and embedded submanifold, which we call a **pair of manifolds**. The blowup $\mathrm{Blup}(M, N)$ of N in M is as a set given by

$$\mathrm{Blup}(M, N) = (M \setminus N) \cup \mathbb{P}, \tag{3.15}$$

i.e. by replacing N with $\mathbb{P} = \mathbb{P}(\nu_N(M))$, the projectivisation of the normal bundle

$$\nu_N(M) = TM|_N / TN$$

of N in M. There is a natural surjective map $p\colon \mathrm{Blup}(M,N) \to M$, called **blowdown map**, given by the fibre bundle projection on $\mathbb{P}$ and by the identity on $M \setminus N$.

However, (3.15) does as written does not carry an obvious smooth structure. There are several equivalent ways to define the blowup as a manifold. Since they are all useful to keep in mind we list them here, but define the blowup using a universal property [Obs21, Proposition 5.30].

Definition 3.3.1 The blowup of N in M is given by a pair of manifolds (B, P) together with a map of pairs $p\colon (B, P) \to (M, N)$ such that

1. $P \subseteq B$ is a submanifold of codimension 1,
2. $p^{-1}(N) = P$ and the normal derivative $\mathrm{d}^N p\colon \nu_P(B) \to \nu_N(M)$ is fibrewisely injective,
3. it satisfies the following universal property: If (X, Y) is another pair of manifolds and $q\colon (X, Y) \to (M, N)$ satisfies the first two conditions, there exists a unique map of pairs $\Phi\colon (X, Y) \to (B, P)$ such that $q = p \circ \Phi$.

The map $p\colon B \to P$ is called the blow-down map.

In accordance to the description in (3.15) we write $\mathrm{Blup}(M, N) = B$ and $\mathbb{P} = P$ for the codimension 1 submanifold. From Definition 3.3.1 we see that if $\operatorname{codim} N = 1$ then $\mathrm{Blup}(M, N) = M$ are isomorphic via the blowdown map, while it hides that the blowdown map is proper (see [AK10, Lemma 2.2] and [Obs21, Proposition 5.34] for a proof), and that the restriction

$$p\big|_{\mathrm{Blup}(M,N)\setminus\mathbb{P}}\colon \mathrm{Blup}(M, N) \setminus \mathbb{P} \to M \setminus N \tag{3.16}$$

is a diffeomorphism.

One way to construct the blowup is via the **deformation to the normal cone** [DS21, Section 4], which also gives insights into the functorial properties of the blowup. As a set the deformation to the normal cone is given by

$$\mathrm{DNC}(M, N) = M \times (\mathbb{R} \setminus \{0\}) \cup (\nu_N(M) \times \{0\}) \tag{3.17}$$

and is endowed with a smooth structure that magnifies normal directions for $\mathbb{R} \ni t \to 0$, which can be understood in terms of smoothness of a $\mathbb{R}^\times$-action given by

$$\lambda.(v, t) = \begin{cases} (v, \lambda^{-1} t) & \text{if } t \neq 0 \\ (\lambda v, 0) & \text{if } t = 0. \end{cases} \tag{3.18}$$

This action is proper and free on $\mathrm{DNC}(M, N) \setminus (N \times \mathbb{R})$. Here we consider $N \times \{0\}$ to sit inside the normal bundle $\nu_N(M)$ as the zero section. The **blowup of N in M** is then given by the quotient

$$\mathrm{Blup}(M, N) = \frac{\mathrm{DNC}(M, N) \setminus (N \times \mathbb{R})}{\mathbb{R}^\times} = M \setminus N \cup \mathbb{P}. \tag{3.19}$$

Note that, if $\operatorname{codim} N = 1$, then $\mathrm{Blup}(M,N) \simeq M$ are isomorphic via the blowdown map. Moreover, the blowdown map is proper (see [AK10, Lemma 2.2] and [Obs21, Proposition 5.34] for a proof), and the restriction

$$p|_{\mathrm{Blup}(M,N)\setminus\mathbb{P}} \colon \mathrm{Blup}(M,N) \setminus \mathbb{P} \to M \setminus N \tag{3.20}$$

is a diffeomorphism.

Remark 3.3.2 Even though the deformation to the normal cone constitutes a functor from the categories of pairs of manifolds to the category of manifolds, the same is not quite true for blowups, see [DS21, Definition 4.8]. If $f\colon (M,N) \to (X,Y)$ is a map of pairs of manifolds, we can only define a blowup of f on

$$\mathrm{Blup}(f)\colon \mathrm{Blup}_f(M,N) := \frac{\mathrm{DNC}(M,N) \setminus \mathrm{DNC}(f)^{-1}(Y \times \mathbb{R})}{\mathbb{R}^\times} \to \mathrm{Blup}(X,Y). \tag{3.21}$$

In particular, $\mathrm{Blup}(f)$ is defined on all of $\mathrm{Blup}(M,N)$ iff $f^{-1}(Y) = N$ and the normal derivative of f is fibrewisely injective.

When performing local computations charts describing the smooth structure of the blowup are most useful [Obs21, Remark 5.29].

Remark 3.3.3 (Charts for $\mathrm{Blup}(M,N)$) Let $N \subseteq M$ be a closed and embedded submanifold of codimension k. Let $(U,(y,x))$ be a submanifold chart, i.e. $U \cap N = \{x = 0\}$, where for simplicity we assume $(y,x)(U) = \mathbb{R}^n$. Then the collection $\{U_i\}_{i=1,\dots,k}$ defined by

$$U_i = p^{-1}(\{x_i \neq 0\}) \cup \{[v]_q \in \mathbb{P}_{U\cap N} : \mathrm{d}x_i(v) \neq 0\} \tag{3.22}$$

yields an open cover of $p^{-1}(U)$ and for $i \in \{1,\dots,k\}$ the maps

$$\Phi_i = (y,\tilde{x})\colon U_i \to \mathbb{R}^n, \tag{3.23}$$

where

$$\tilde{x}(\xi) = \begin{cases} \big(\frac{\mathrm{d}x_1(v)}{\mathrm{d}x_i(v)},\dots,\frac{\mathrm{d}x_{i-1}(v)}{\mathrm{d}x_i(v)},0,\frac{\mathrm{d}x_{i+1}(v)}{\mathrm{d}x_i(v)},\dots\frac{\mathrm{d}x_k(v)}{\mathrm{d}x_i(v)}\big) & \text{if } \xi = [v]_{p(\xi)}, \\ \big(\frac{x_1(p(\xi))}{x_i(p(\xi))},\dots,\frac{x_{i-1}(p(\xi))}{x_i(p(\xi))},x_i(p(\xi)),\frac{x_{i+1}(p(\xi))}{x_i(p(\xi))},\dots\frac{x_k(p(\xi))}{x_i(p(\xi))}\big) & \text{otherwise,} \end{cases}$$

yield charts for $\mathrm{Blup}(U, U\cap N) \subseteq \mathrm{Blup}(M,N)$. The blowdown map in these coordinates reads

$$p(y,\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_k) = (y,\tilde{x}_i\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_i\tilde{x}_k). \tag{3.24}$$

Lemma 3.3.4 *Let $N \subseteq M$ be a closed and embedded submanifold, $X \in \Gamma(TM)$, and write $p\colon \mathrm{Blup}(M,N) \to M$ for the blowdown map. Then there exists a vector field $\tilde{X} \in \Gamma(T\mathrm{Blup}(M,N))$ with $\tilde{X} \sim_p X$ if and only if $X \in \Gamma(TM,TN)$, i.e. X is tangent to N. In that case, $\tilde{X}$ is unique and tangent to $\mathbb{P} \subseteq \mathrm{Blup}(M,N)$.*

Proof. See also [LGLR24, Proposition 1.5.40]. First suppose that $\tilde{X}$ exists. Note that for $[v] \in \mathbb{P}$ we have $\operatorname{im}(T_{[v]}p) = T_{p([v])}N \oplus \mathbb{R}v$, making use of a tubular neighbourhood. Then, for $q \in N$,

$$\bigcap_{[v]\in\mathbb{P}_q} \operatorname{im}(T_{[v]}p) = T_qN,$$

so $X \in \Gamma(TM, TN)$ follows immediately. Conversely let $X \in \Gamma(TM, TN)$ be given. Take an adapted chart $(U, (y, x))$ of N in M with $N \cap U = \{x = 0\}$. Locally, X is given by

$$X\big|_U = \sum_\alpha f_\alpha \frac{\partial}{\partial y_\alpha} + \sum_j g_j \frac{\partial}{\partial x_j}$$

for $f_\alpha, g_j \in \mathscr{C}^\infty(M)$, where all g_j vanish on N. Fix $i \in \{1, \ldots, \operatorname{codim} N\}$ and consider the chart $(U_i, \Phi_i = (y, \tilde{x}))$ of the blowup adapted to $\mathbb{P}$ (given by $\tilde{x}_i = 0$). Then on $U_i \setminus \mathbb{P}$ we have

$$\tilde{X}\big|_{U_i} = \sum_\alpha p^* f_\alpha \frac{\partial}{\partial y_\alpha} + p^* g_i \frac{\partial}{\partial \tilde{x}_i} + \sum_{j\neq i} \frac{1}{\tilde{x}_i}(p^* g_j - p^* g_i \tilde{x}_j)\frac{\partial}{\partial \tilde{x}_j}, \tag{3.25}$$

Since for all $j \in \{1, \ldots, \operatorname{codim} N\}$ we have $g_j\big|_N = 0$, $p^* g_j\big|_{\mathbb{P}} = 0$ follows. Thus, there are functions $\tilde{g}_j \in \mathscr{C}^\infty(U_i)$ such that $p^* g_j = \tilde{x}_i \tilde{g}_j$. Inserting this in $(*)$ shows that $p^* X\big|_{U\setminus N}$ extends smoothly to a vector field on U_i, hence $p^* X\big|_{M\setminus N}$ extends smoothly to $\mathbb{P}$. ☐

The final point of view we want to take is locally around the submanifold N in a tubular neighbourhood. Using gluing, this is enough to define the blowup even globally, see e.g. [Mik97, Section 2]. Moreover, it implies that every tubular neighbourhood of N in M induces a tubular neighbourhood of $\mathbb{P}$ in $\operatorname{Blup}(M, N)$, see Corollary 3.3.6.

Remark 3.3.5 Let $E \to N$ be a vector bundle and view N as a subset of E via the image of the zero section. Then

$$\operatorname{Blup}(E, N) \simeq \mathbb{L}(E), \tag{3.26}$$

where $\mathbb{L}(E)$ is the tautological line bundle over the projectivisation of E, i.e. it is the line bundle over $\mathbb{P}$ with fibres given by

$$\mathbb{L}(E)_{[v]} = \{v' \in E \colon v' = \lambda v \text{ for some } \lambda \in \mathbb{R}\} \tag{3.27}$$

for $v \in E \setminus N$. In this case, the blowdown map is given by $v'_{[v]} \mapsto v'$.

Thus, every tubular neighbourhood of N in M induces a tubular neighbourhood of $\mathbb{P}$ in $\operatorname{Blup}(M, N)$. We show that the respective Euler vector fields are related by the blowdown map.

Corollary 3.3.6 *Let $E \to N$ be a vector bundle with Euler vector field $\xi \in \Gamma(TE)$. Then the lift of ξ is the Euler vector field of the line bundle $\mathbb{L} = \operatorname{Blup}(E, N) \to \mathbb{P}$.*

Proof. Lemma 3.3.4 implies that ξ lifts to the blowup. Since the blowdown map is a vector bundle morphism and a diffeomorphism on $E \setminus N$, we can compare the flows of the vector fields there to see that the lift is indeed the Euler vector field. $\square$

3.3.2 Blowups of Lie algebroids

Starting from a Lie algebroid $A \Rightarrow M$ and a closed and embedded Lie subalgebroid $B \Rightarrow N$, the blowup $\mathrm{Blup}(A, B)$ will in general not carry a natural structure of a Lie algebroid. In fact, if $\mathrm{rank}\,(B) < \mathrm{rank}\,(A)$, by Remark 3.3.2 already the vector bundle projection $\pi\colon A \to M$ of A will not lift to the blowup. Instead, one has to consider $\mathrm{Blup}_\pi(A, B)$.

Note that there is a corresponding construction for Lie groupoids [DS21], see also [Obs21]. The projective blowup construction has been used to desingularise proper groupoids [PTW21, Wan18], and in the context of Lie algebroids blowing up has been shown to recover interesting Lie algebroids (a construction called **elementary modification** in [GL13, Definition 2.11]), like log- or scattering tangent bundles (see also [Kla17, Lan21], and Example 3.3.12).

We start by blowing up vector subbundles, then anchored subbundles, and finally Lie subalgebroids.

Lemma 3.3.7 *Let $\pi\colon A \to M$ be a vector bundle and $B \to N$ a closed and embedded subbundle. Then $\mathrm{Blup}(\pi)\colon \mathrm{Blup}_\pi(A, B) \to \mathrm{Blup}(M, N)$ is a vector bundle with sections given by the $\mathscr{C}^\infty(\mathrm{Blup}(M, N))$-span of*

$$\{\mathrm{Blup}(s)\colon \mathrm{Blup}(M, N) \to \mathrm{Blup}_\pi(A, B) : s \in \Gamma(A, B)\}, \tag{3.28}$$

considering $s \in \Gamma(A, B)$, i.e. $s\big|_N \in \Gamma(B)$, as a map of pairs $(M, N) \to (A, B)$.

Proof. See [Obs21, Section 5.4.2]. Note that, since s is a section,

$$\mathrm{Blup}(s)\colon \mathrm{Blup}_s(M, N) \to \mathrm{Blup}(A, B)$$

is actually defined on all of $\mathrm{Blup}(M, N)$ and maps into $\mathrm{Blup}_\pi(A, B)$. $\square$

From Lemma 3.3.7 we can immediately write down local frames for the blowup.

Remark 3.3.8 (Local frames for $\mathrm{Blup}_\pi(A, B)$) Let $B \to N$ be a closed and embedded vector subbundle of corank k of $\pi\colon A \to M$. Let $(U, (y, x))$ be a submanifold chart of N in M, i.e. $U \cap N = \{x = 0\}$. Moreover, let $\{e_1, \ldots, e_k, f_1, \ldots, f_{\mathrm{rank}\, B}\}$ be a local frame of $A\big|_U$ adapted to B, meaning $\{f_1\big|_N, \ldots f_{\mathrm{rank}\, B}\big|_N\}$ is a local frame for $B\big|_{U \cap N}$. Then the collection

$$\{\mathrm{Blup}(x_i e_1), \ldots, \mathrm{Blup}(x_i e_k), \mathrm{Blup}(f_1), \ldots, \mathrm{Blup}(f_{\mathrm{rank}\, B})\} \tag{3.29}$$

yields a local frame for $\mathrm{Blup}_\pi(A, B)\big|_{U_i}$.

Lemma 3.3.9 *Let $B \to N$ be a closed and embedded subbundle of a vector bundle $\pi\colon A \to M$ and let $p\colon \mathrm{Blup}(M, N) \to M$ denote the blowdown map of the base.*

1. *As vector bundles,* $\mathrm{Blup}_\pi(A, A|_N) = p^\sharp A$ *is given by the pullback bundle. Under this identification we have* $p_A = p^\sharp$*.*
2. *As vector bundles,* $\mathrm{Blup}_{\pi_{p^\sharp A}}(p^\sharp A, p^\sharp B) = \mathrm{Blup}_\pi(A, B)$*.*

Proof. Both statements follow from Remark 3.3.8 using their respective local frames. ☐

Given a map of pairs of vector bundles $\Phi\colon (A, B) \to (E, F)$ over a base map $\phi\colon (M, N) \to (X, Y)$ we obtain an induced map of vector bundles

$$\mathrm{Blup}(\Phi)\colon \mathrm{Blup}_{\pi_A}(A, B)|_{\mathrm{Blup}_\phi(M,N)} \to \mathrm{Blup}_{\pi_E}(E, F). \tag{3.30}$$

Next, we can include an anchor map into the construction, i.e. a vector bundle morphism $\sharp\colon A \to TM$ over the identity.

Lemma 3.3.10 *Let* $(\pi\colon A \to M, \sharp)$ *be an anchored vector bundle and* $B \to N$ *a closed and embedded subbundle with* $\sharp(B) \subseteq TN$*. Then there is an induced anchor on* $\mathrm{Blup}_\pi(A, B)$ *which on blowup sections is given by*

$$\sharp_{\mathrm{Blup}}(\mathrm{Blup}(s)) = \widetilde{\sharp(s)}, \tag{3.31}$$

where $s \in \Gamma(A, B)$ *and* $\widetilde{\sharp(s)}$ *is the extension from Lemma 3.3.4. In particular, the blowdown map is a morphism of anchored vector bundles.*

Alternatively, one can blow up the map of pairs $\sharp\colon (A, B) \to (TM, TN)$ and consider the composition

$$\sharp_{\mathrm{Blup}}\colon \mathrm{Blup}_\pi(A, B) \to \mathrm{Blup}_{\pi_{TM}}(TM, TN) \to T\mathrm{Blup}(M, N), \tag{3.32}$$

where the second map is given by the identity in fibres over $M \setminus N$ and by the differential of the projection $\nu_N(M)\setminus 0 \to \mathbb{P}$ over the projective bundle [Obs21, Proposition 5.58]. Lastly, the blowup of Lie algebroids can be defined using a universal property, analog to Definition 3.3.1. For this, we call (A, B) a **pair of Lie algebroids**, where $B \Rightarrow N$ is a closed and embedded Lie subalgebroid of $A \Rightarrow M$.

Definition 3.3.11 Let (A, B) be a pair of Lie algebroids. The **blowup of** B **in** A is a pair of Lie algebroids $(\tilde{A}, \tilde{B})$ together with a morphism $p_A\colon (\tilde{A}, \tilde{B}) \to (A, B)$ of pairs of Lie algebroids such that

1. $\tilde{B}$ is a closed and embedded Lie subalgebroid of full rank over a base of codimension 1,
2. $p_A^{-1}(B) = \tilde{B}$ and the normal derivative $\mathrm{d}^N p_A\colon \nu_{\tilde{B}}(\tilde{A}) \to \nu_A(B)$ is fibrewisely injective,
3. it satisfies the following universal property: If $F \subseteq E$ is another closed and embedded Lie subalgebroid and $\phi\colon (E, F) \to (A, B)$ a morphism of pairs of Lie algebroids such that the first two conditions are satisfied, there exists a unique morphism of pairs of Lie algebroids $\Phi\colon (E, F) \to (\tilde{A}, \tilde{B})$ such that $\phi = p_A \circ \Phi$.

Note that the blowups of submanifolds and (anchored) vector bundles satisfy similar universal properties, see e.g. [Obs21, Proposition 5.30] for the blowup of submanifolds.

One can obtain an explicit model for the blowup of Lie algebroids by equipping $\mathrm{Blup}_\pi(A, B)$ with the anchor from Lemma 3.3.10 and bracket on blowup sections given by

$$[\mathrm{Blup}(a), \mathrm{Blup}(b)] = \mathrm{Blup}([a, b]) \tag{3.33}$$

for $a, b \in \Gamma(A, B)$. Together with the Leibniz rule, (3.33) defines the bracket uniquely as these sections generate $\Gamma(\mathrm{Blup}_\pi(A, B))$. Then the blowdown map $p_A \colon \mathrm{Blup}_\pi(A, B) \to A$ becomes a Lie algebroid morphism over the blowdown map of the base $p \colon \mathrm{Blup}(M, N) \to M$. Proving that this construction satisfies the universal property is analog to the case of blowups of manifolds, for which it is given in [Obs21, Proposition 5.30].

Example 3.3.12 The construction of **Elementary modifications** [Kla17, GL13, Lan21] can e expressed in terms of blowups. If $N \subseteq M$ has $\operatorname{codim}(N) = 1$, the blowdown map is a diffeomorphism. In particular, when blowing up the base manifold does not change, and in this case the sections of the Lie algebroid blowup are precisely given by

$$\Gamma(\mathrm{Blup}_\pi(A, B)) = \Gamma(A, B) \tag{3.34}$$

by Lemma 3.3.7. Examples of these modifications are

1. the log-tangent bundle $T^b_N M = \mathrm{Blup}_{\pi_{TM}}(TM, TN)$;
2. the scattering tangent bundle $\mathrm{Blup}_{\pi_{T^b_N M}}(T^b_N M, 0_N)$.

3.4 The blowdown map in cohomology

In this section we study the blowdown map in cohomology (3.1). Since, given a Lie subalgebroid B of A, we only consider the Lie algebroid blowup, we drop the subscript π from now on.

We first consider the pullback by the blowdown map

$$p_A^* \colon \Omega^\bullet(A) \to \Omega^\bullet(\mathrm{Blup}(A, B)) \tag{3.35}$$

on the level of forms. Given a closed and embedded Lie subalgebroid $B \Rightarrow N$ of $A \Rightarrow M$ the pullback (3.35) will not be surjective unless both $N \subseteq M$ is of codimension 1 and $B = A_N$. However, if we restrict to forms that are flat along N and $\mathbb{P} = p^{-1}(N)$, respectively, p_A^* becomes an isomorphism.

Recall that for a vector bundle $E \to M$ and a submanifold $N \subseteq M$ we denote the submodule of **flat sections of E along N** by

$$\Gamma_N(E) = \bigcap_{k \in \mathbb{N}} \mathcal{I}_N^k \Gamma(E), \tag{3.36}$$

where $\mathcal{I}_N \subseteq \mathscr{C}^\infty(M)$ is the vanishing ideal of N. Classes in the quotient

$$\mathscr{J}_N^\infty \Gamma(E) = \frac{\Gamma(E)}{\Gamma_N(E)} \tag{3.37}$$

are then called **∞-jets of sections along** N. The result of this section, which shows that the blowdown map in cohomology only depends on local data, then reads the following.

Theorem 3.4.1 *Let $B \Rightarrow N$ be an embedded Lie subalgebroid of $A \Rightarrow N$ over a closed base $N \subseteq M$.*

1. *The induced map on flat forms*
$$p_A^* \colon \Omega_N^\bullet(A) \to \Omega_{\mathbb{P}}^\bullet(\mathrm{Blup}(A, B)) \tag{3.38}$$
is an isomorphism of cochain complexes.

2. *By part 1 we obtain an isomorphism of cochain complexes*
$$\frac{\Omega^\bullet(\mathrm{Blup}(A, B))}{p_A^*\Omega^\bullet(A)} \cong \frac{\mathscr{J}_{\mathbb{P}}^\infty\Omega^\bullet(\mathrm{Blup}(A, B))}{p_A^*\mathscr{J}_N^\infty\Omega^\bullet(A)}. \tag{3.39}$$
In particular, in the long exact sequence
$$\ldots \longrightarrow \mathrm{H}^\bullet(A) \xrightarrow{p_A^*} \mathrm{H}^\bullet(\mathrm{Blup}(A, B)) \xrightarrow{f} \mathrm{H}^\bullet\left(\frac{\Omega^\bullet(\mathrm{Blup}(A, B))}{p_A^*\Omega^\bullet(A)}\right) \longrightarrow \ldots \tag{3.40}$$
the map f only depends on local data around N and $\mathbb{P}$, e.g. if $\iota \colon U \hookrightarrow \mathrm{Blup}(M, N)$ is an open neighbourhood of $\mathbb{P}$, then $f = f_U \circ \mathrm{H}(\iota^*)$. Here, $\mathrm{H}(\iota^*) \colon \mathrm{H}^\bullet(\mathrm{Blup}(A, B)) \to \mathrm{H}^\bullet(\mathrm{Blup}(A, B)\big|_U)$ denotes the map induced in cohomology by the inclusion and f_U is the map corresponding to f in the long exact sequence (3.40) *for $A\big|_{p(U)}$.*

To show the first part of Theorem 3.4.1, we use that a form on A that is flat along N can be written as a locally finite sum of $\wedge$-products of flat functions and flat 1-forms. To prove this in Lemma 3.4.3, we need the following [Nag73, Theorem 1].

Lemma 3.4.2 *Let $U \subseteq \mathbb{R}^n$ be open and $F \subseteq U$ be closed. Let $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathscr{C}_F^\infty(U)$ be any countable collection of functions which are flat along F. Then there exist $\{g_i\}_{i\in\mathbb{N}} \subseteq \mathscr{C}_F^\infty(U)$ and $h \in \mathscr{C}_F^\infty(U)$ such that*

1. *$h(x) > 0$ if $x \in U \setminus F$,*
2. *$f_i = hg_i$ for all $i \in \mathbb{N}$.*

Lemma 3.4.3 *Let $E \to M$ be a vector bundle and $N \subseteq M$ a closed and embedded submanifold. Then any $\omega \in \Gamma_N(\Lambda^\bullet E)$ can be written as a locally finite sum of $\wedge$-products of elements in $\mathscr{C}_N^\infty(M) \oplus \Gamma_N(E)$.*

Proof. Let $(U_\alpha, x_\alpha)_\alpha$ be an atlas of submanifold charts, $\{e_\alpha^1, \ldots, e_\alpha^\ell\}$ a local frame of $E\big|_{U_\alpha}$ and $\omega \in \Gamma_N(\Lambda^k E)$ for some $k > 1$. Let $\{\chi_\alpha\}_\alpha$ be a partition of unity of order $k+1$ subordinate to the open cover $\{U_\alpha\}_\alpha$, i.e. $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$ for all α and $\sum_\alpha \chi_\alpha^{k+1} = 1$. Locally,
$$\omega\big|_{U_\alpha} = \sum_{j_1,\ldots,j_k} f_{\alpha,j_1,\ldots,j_k} e_\alpha^{j_1} \wedge \cdots \wedge e_\alpha^{j_k}$$

in the chart (U_α, x_α). Flatness of ω along N means flatness of all coefficient functions $f_{\alpha,j_1,\dots,j_k}$ along $U_\alpha \cap N$. In particular, $\chi_\alpha f_{\alpha,j_1,\dots,j_k}$ is flat along $U_\alpha \cap N$ and by applying Lemma 3.4.2 k times we find $g_{1,\alpha,j_1,\dots,j_k}, \dots, g_{k,\alpha,j_1,\dots,j_k} \in \mathscr{C}^\infty_{U_\alpha \cap N}(U_\alpha)$ such that

$$g_{1,\alpha,j_1,\dots,j_k} \cdots g_{k,\alpha,j_1,\dots,j_k} = \chi_\alpha f_{\alpha,j_1,\dots,j_k}.$$

Thus, we can decompose

$$\begin{aligned} \omega = \sum_\alpha \chi_\alpha^{k+1} \omega &= \sum_{\alpha,j_1,\dots,j_k} \chi_\alpha^k \chi_\alpha f_{\alpha,j_1,\dots,j_k} e_\alpha^{j_1} \wedge \dots \wedge e_\alpha^{j_k} \\ &= \sum_{\alpha,j_1,\dots,j_k} \chi_\alpha^k g_{1,\alpha,j_1,\dots,j_k} \cdots g_{k,\alpha,j_1,\dots,j_k} e_\alpha^{j_1} \wedge \dots \wedge e_\alpha^{j_k} \\ &= \sum_{\alpha,j_1,\dots,j_k} \big(\chi_\alpha g_{1,\alpha,j_1,\dots,j_k} e_\alpha^{j_1}\big) \wedge \dots \wedge \big(\chi_\alpha g_{k,\alpha,j_1,\dots,j_k} e_\alpha^{j_k}\big), \end{aligned}$$

which proves the claim as $\chi_\alpha g_{\ell,\alpha,j_1,\dots,j_k} e_\alpha^{j_\ell}$ is a well-defined 1-form on all of M, which is flat along N. ☐

Using Lemma 3.4.3 we can show Theorem 3.4.1. Since the first part is a statement about sections of vector bundles and does not involve the additional structures provided by a Lie algebroid, we formulate and prove it in a separate lemma.

Lemma 3.4.4 *Let $F \to N$ be a closed and embedded vector subbundle of $E \to M$. Then the pullback by the blowdown map $p_E^*\colon \Gamma(E^*) \to \Gamma(\mathrm{Blup}(E,F)^*)$ restricts to an isomorphism*

$$p_E^*\colon \Gamma_N(E^*) \xrightarrow{\sim} \Gamma_{\mathbb{P}}(\mathrm{Blup}(E,F)^*). \tag{3.41}$$

Proof. The proof consists of two steps. First, we show that $p^*\colon \mathscr{C}^\infty_N(M) \to \mathscr{C}^\infty_{\mathbb{P}}(\mathrm{Blup}(M,N))$ is an isomorphism (which corresponds to $E = M \times \mathbb{R}$ and $F = N \times \mathbb{R}$). Then we can prove the statement for general vector bundles. It is clear that p^* factors to flat functions (as $p^*\mathcal{I}_N \subseteq \mathcal{I}_{\mathbb{P}}$) and is injective. Thus, let $f \in \mathscr{C}^\infty_{\mathbb{P}}(\mathrm{Blup}(M,N))$ be given. Since $f\big|_{\mathbb{P}} = 0$ we find a function $\tilde{f} \in \mathscr{C}^\infty(M)$ with $p^*\tilde{f} = f$, which is automatically continuous. To show that it is smooth and flat along N, we use the charts from Remark 3.3.3. First we note that differentiation along N is just differentiating along the base coordinates of $\mathbb{P}$, hence there is nothing to check. Thus, we can assume that N is a point and, since differentiation is local, $M = \mathbb{R}^n$. As the computations in arbitrary dimensions are the same as in two with more bookkeeping, suppose $n = 2$.

In the chart Φ_1 the blowdown map maps

$$p(\Phi_1^{-1}(x_1, x_2)) = (x_1, x_1 x_2).$$

We use (x_1, x_2) for the coordinates on $U_1 \subseteq \mathrm{Blup}(\mathbb{R}^2, \{0\})$ and (x, y) on $\mathbb{R}^2$. We show inductively that for any $\alpha \in \mathbb{N}$ and $\beta \le k$ the derivative $\frac{\partial^\beta}{\partial x^\beta} \frac{\partial^{\alpha-\beta}}{\partial y^{\alpha-\beta}} \tilde{f}$

pulls back to a function in $\mathscr{C}^\infty_{\mathbb{P}}(\mathrm{Blup}(M,N))$ and thus extends continuously to zero. Regarding the differential of the blowdown map we find

$$\begin{aligned} T_{(x_1,x_2)}p(\partial_{x_1}) &= \partial_x + x_2\partial_y = \partial_x + \frac{y}{x}\partial_y \\ T_{(x_1,x_2)}p(\partial_{x_2}) &= x_1\partial_y = x\partial_y. \end{aligned}$$

In other words, away from the origin in $p(U_1)$ we have for the pullback with p that

$$\begin{aligned} p^*\partial_x &= \partial_{x_1} - \frac{x_2}{x_1}\partial_{x_2} \\ p^*\partial_y &= \frac{1}{x_1}\partial_{x_2}. \end{aligned} \tag{3.42}$$

These are singular vector fields, but since f is flat along $\mathbb{P}$ we find $g \in \mathscr{C}^\infty_{\mathbb{P}\cap U_1}(U_1)$ such that $f = x_1 g$. Hence, we can apply them to f, yielding a function that is again flat along $\mathbb{P}$. The same computation for (U_2, Φ_2) shows that the first derivatives of $\tilde{f}$ indeed pull back to flat functions as claimed. But then the same argument also shows the step $k \to k+1$.

For the general vector bundle blowup, let $(U,(x,y))$ be a submanifold chart of N such that $U \cap N = \{x = 0\}$, and $\{e_1, \dots, e_{\mathrm{corank} F}, f_1, \dots, f_{\mathrm{rank}\, F}\}$ be a local frame for E_U such that the collection $\{f_\alpha\}_\alpha$ restricts to a frame of F over $U \cap N$. Recall that for $U_i \subseteq \mathrm{Blup}(M,N)$, $i = 1, \dots, \operatorname{codim} N$, a local frame for $\mathrm{Blup}(E,F)\big|_{U_i}$ is given by

$$\{\mathrm{Blup}(x_i e_1), \dots, \mathrm{Blup}(x_i e_{\mathrm{corank} F}), \mathrm{Blup}(f_1), \dots, \mathrm{Blup}(f_{\mathrm{rank}\, F})\}, \tag{$*$}$$

see Remark 3.3.8. For the pullbacks of the dual frames of E_U to U_i one immediately finds

$$\begin{aligned} p_E^* f^\alpha(\mathrm{Blup}(f_\alpha)) &= 1, \\ p_E^* e^k(\mathrm{Blup}(x_i e_k)) &= x_i, \end{aligned}$$

while all other pairings are zero. Writing the dual frame of $(*)$ as

$$\{\mathrm{Blup}(x_i e_1)^*, \dots, \mathrm{Blup}(x_i e_{\mathrm{corank} B})^*, \mathrm{Blup}(f_1)^*, \dots, \mathrm{Blup}(f_{\mathrm{rank}\, B})^*\},$$

any $\hat{\omega} \in \Gamma_{\mathbb{P}}(\mathrm{Blup}(E,F))$ can locally be written as $\hat{\omega}\big|_{U_i} = \sum_k \hat{\omega}^k \mathrm{Blup}(x_i e_k)^* + \sum_\alpha \hat{\omega}^\alpha \mathrm{Blup}(f_\alpha)^*$. But since $\hat{\omega}$ is flat along $\mathbb{P} \cap U_i$, the functions $\frac{\hat{\omega}^k}{x_i}$ are still well-defined on U_i and flat along $\mathbb{P} \cap U_i$. On $p(U_i) \setminus N$, we can define the 1-form

$$\omega_i = \sum_\alpha (p_E)_*(\hat{\omega}^\alpha) f^\alpha + \sum_k (p_E)_*\big(\tfrac{\hat{\omega}^k}{x_i}\big) e^k.$$

Since $p_E^* \omega_i = \hat{\omega}\big|_{U_i \setminus \mathbb{P}}$, we have $\omega_i = \omega_j$ for $i, j \in \{1, \dots, \operatorname{codim} N\}$ whenever their domain of definition overlap. Thus, gluing gives $\omega \in \Omega^1(U \setminus N)$, which extends flatly to $N \cap U$ and satisfies $p_E^* \omega = \hat{\omega}\big|_{p^{-1}(U)}$. □

Proof of Theorem 3.4.1. The first part follows from Lemma 3.4.3 and 3.4.4.

For the second part consider the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^\bullet_N(A) \arrow[r, "p_A^*"] \arrow[d] & \Omega^\bullet_{\mathbb{P}}(\mathrm{Blup}(A,B)) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^\bullet(A) \arrow[r, "p_A^*"] \arrow[d] & \Omega^\bullet(\mathrm{Blup}(A,B)) \arrow[r] \arrow[d] & \frac{\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\Omega^\bullet(A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{J}_N^\infty\Omega^\bullet(A) \arrow[r, "p_A^*"] \arrow[d] & \mathscr{J}_{\mathbb{P}}^\infty\Omega^\bullet(\mathrm{Blup}(A,B)) \arrow[r] \arrow[d] & \frac{\mathscr{J}_{\mathbb{P}}^\infty\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\mathscr{J}_N^\infty\Omega^\bullet(A)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Exactness of the first and second column is clear. Regarding exactness of the second and third row note that p_A is a diffeomorphism on $M \setminus N$, implying injectivity of p_A^*. From the first part we get exactness of the first row, and thus

$$\frac{\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\Omega^\bullet(A)} \cong \frac{\mathscr{J}_{\mathbb{P}}^\infty\Omega^\bullet(\mathrm{Blup}(A,B))}{p_A^*\mathscr{J}_N^\infty\Omega^\bullet(A)}$$

by the 3×3-Lemma [Mac63, Chapter 2, Lemma 5.1]. The remaining claim follows from the local nature of jet spaces. ☐

3.5 Blowups of transversals

One class of Lie subalgebroids of $A \Rightarrow M$ is given by transverse submanifolds $\iota\colon N \hookrightarrow M$, i.e. embedded submanifolds such that the inclusion is transverse to the anchor. Then $\sharp^{-1}(TN) = \iota^! A \subseteq A$ is a Lie subalgebroid, see Section 1.1.2. An example of such is e.g. $TN \subseteq TM$ for any closed submanifold $N \subseteq M$ as a Lie subalgebroid of the tangent Lie algebroid. We compute the cohomology of $\mathrm{Blup}(A, \iota^! A)$ in Corollary 3.5.6 after characterising the blowdown map in cohomology in this setting in Theorem 3.5.4.

A property of transversals we will use is that they admit simple normal forms [BLM16, Section 4].

Theorem 3.5.1 *Let $A \Rightarrow M$ be a Lie algebroid and $\iota\colon N \hookrightarrow M$ a transversal. Then there exists a tubular neighbourhood* $\mathrm{pr}\colon E \to N$ *of N in M such that*

$$A_E \cong \mathrm{pr}^! \iota^! A \tag{3.43}$$

are isomorphic as Lie algebroids.

Moreover, any such isomorphism $A_E \cong \mathrm{pr}^! \iota^! A$ induces an isomorphism

$$\mathrm{H}^\bullet(A_E) = \mathrm{H}^\bullet(\iota^! A) \tag{3.44}$$

by means of the restriction map, see [Fre19, Theorem 2]. The proof in [Fre19] utilises [Cra00, Theorem 2], a spectral sequence argument, see also [MS24, Section 5.2].

According to Theorem 3.4.1 it is enough to understand the blowdown map in cohomology in a neighbourhood of N, thus, we can utilise Theorem 3.5.1 to calculate $\mathrm{H}^\bullet(\mathrm{Blup}(A, \iota^! A))$. To formulate the result, we need two definitions.

Definition 3.5.2 Let $A \Rightarrow M$ be a Lie algebroid and $B \Rightarrow N$ a Lie subalgebroid of A.

1. A vector field $X \in \Gamma(TM)$ is called **Euler-like along** N if it is complete and there is a tubular neighbourhood embedding $\phi\colon E \to U$ of a vector bundle $E \to N$ onto an open neighbourhood U of N in M such that $\phi^* X$ is the Euler vector field of E [BLM16, Definition 2.6].
2. A section $a \in \Gamma(A, B)$ is called **Euler-like along** B if $\sharp(A)$ is Euler-like along N and it induces the trivial inner derivation

$$[a\big|_N, \cdot]_B = 0\colon \Gamma(B) \to \Gamma(B). \tag{3.45}$$

From [BLM16, Lemma 3.9] we know that, given a transversal N, we can always find Euler-like sections $a \in \Gamma(A)$ along B with $a\big|_N = 0$. Note that this definition of an Euler-like section of a Lie algebroid differs from the one given in [BBLM20] in the way that we do not require $a\big|_N$ itself to vanish. The reason behind this is that, for an isomorphism like (3.44) to hold, the existence of an Euler-like section in our sense is enough, one does not necessarily need a trivialisation $A_E \cong \mathrm{pr}^! \iota^! A$ of Lie algebroids, see [MS24, Theorem 3.27].

Moreover, to formulate Theorem 3.5.4 we need the notion of compact vertical cohomology.

Definition 3.5.3 Let $\pi\colon E \to M$ be a fibre bundle and $A \Rightarrow E$ a Lie algebroid. The subcomplex of forms of A compactly supported in vertical directions is defined as

$$\Omega^\bullet_{\mathrm{cv}}(A) = \{\omega \in \Omega^\bullet(A) \colon \mathrm{supp}(\omega) \cap \pi^{-1}(K) \text{ is compact for all compact } K \subseteq M\}. \tag{3.46}$$

Since the differential is a local operator, (3.46) is indeed a subcomplex, with cohomology denoted by $\mathrm{H}^\bullet_{\mathrm{cv}}(A)$, called the **compact vertical cohomology** (see e.g. [BT82]). Of course, the same notion can be defined also for forms with coefficients in some representation. Regarding the cohomology of the blowup of transversals, we obtain the following results.

Theorem 3.5.4 *Let $\iota\colon N \hookrightarrow M$ be a closed transversal of a Lie algebroid $A \Rightarrow M$. Denote the blowdown map by $p_A\colon \mathrm{Blup}(A, \iota^! A) \to A$, the blowdown map of the base by $p\colon \mathrm{Blup}(M, N) \to M$, and the projection of the projective bundle by $\pi_{\mathbb{P}}\colon \mathbb{P} \to N$.*

1. *We have (canonical) isomorphisms*

$$\mathrm{H}^\bullet(\mathrm{Blup}(A, \iota^! A)) \cong \mathrm{H}^\bullet(\mathrm{Blup}(p^! A, \pi_{\mathbb{P}}^! A)) \tag{3.47}$$

$$\cong \mathrm{H}^\bullet(p^! A) \oplus \mathrm{H}^{\bullet-1}(\pi_{\mathbb{P}}^! A). \tag{3.48}$$

   Under the identification (3.48)*, the blowdown map* p_A^* *in cohomology becomes* $(p^!)^* \colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(p^! A)$.

2. *If* codim N *is odd, then*

$$(p^!)^* \colon \mathrm{H}^\bullet(A) \xrightarrow{\sim} \mathrm{H}^\bullet(p^! A). \tag{3.49}$$

3. *If* codim N *is even, then any section* $a \in \Gamma(A)$ *Euler-like along* $\iota^! A$ *with* $a\big|_N = 0$ *and corresponding tubular neighbourhood* $E \to N$ *in* M *gives rise to a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^\bullet(A) \xrightarrow{(p^!)^*} \mathrm{H}^\bullet(p^! A) \xrightarrow{(p^!)_*} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A\big|_E) \xrightarrow{i} \mathrm{H}^{\bullet+1}(A) \longrightarrow \ldots \tag{3.50}$$

   where $(p^!)_*$ *first restricts a form to* $\mathbb{P}$*, fibre integrates and applies the Thom isomorphism (see Lemma 3.8.9), and* $i([\omega]) = [\omega]$ *for any* $[\omega] \in \mathrm{H}^\bullet_{\mathrm{cv}}(A\big|_E)$.

Note that the choice of Euler-like section in the third part only affects the tubular neighbourhood, i.e. two Euler-like sections inducing the same tubular neighbourhood lead to the same long exact sequence (3.50).

Theorem 3.5.4 characterises the blowdown map in cohomology completely. Note that the reason for the distinction between even and odd codimension lies in the de Rham cohomology of the projective spaces that constitute the fibres of $\mathbb{P} \to N$, see also Theorem 3.5.5. In the case of odd codimension, the cohomology is trivial in all but 0-th degree, leading to the simplified form. We prove Theorem 3.5.4 in two steps. First we prove the case of codim $N = 1$ in Section 3.5.1 and then complete the proof in Section 3.5.3.

From Equation (3.48) we see that the only missing ingredient to compute the cohomology of the blowup of a transversal is computing $\mathrm{H}^\bullet(\pi_{\mathbb{P}}^! A)$.

Theorem 3.5.5 *Let* $B \Rightarrow N$ *be a Lie algebroid. Let* $E \to N$ *be a vector bundle of rank* k*,* $o(E) \to N$ *its orientation bundle, and* $e \in \mathrm{H}^k(N, o(E))$ *its Euler class. Let* $\pi_{\mathbb{P}} \colon \mathbb{P} \to N$ *denote the projectivisation of* E*.*

1. *If* k *is odd, we have an isomorphism*

$$(\pi_{\mathbb{P}}^!)^* \colon \mathrm{H}^\bullet(B) \xrightarrow{\sim} \mathrm{H}^\bullet(\pi_{\mathbb{P}}^! B). \tag{3.51}$$

2. *If* k *is even, there is a Gysin-like long exact sequence*

$$\ldots \to \mathrm{H}^\bullet(B) \xrightarrow{(\pi_{\mathbb{P}}^!)^*} \mathrm{H}^\bullet(\pi_{\mathbb{P}}^! B) \xrightarrow{(\pi_{\mathbb{P}}^!)_*} \mathrm{H}^{\bullet-(k-1)}(B, o(E)) \xrightarrow{\wedge \sharp^* e} \mathrm{H}^{\bullet+1}(B) \to \ldots \tag{3.52}$$

   Here, $(\pi_{\mathbb{P}}^!)_*$ *denotes fibre integration.*

Proof. Let N be of odd codimension. In this case, the fibres of the projective bundle $\pi_{\mathbb{P}}\colon \mathbb{P} \to N$ are projective spaces of even dimension, which have nontrivial de Rham cohomology only in degree 0. By the Serre-Leray spectral sequence for Lie algebroids [MS24, Corollary 5.8], we immediately obtain

$$(\pi_{\mathbb{P}}^{!})^{*}\colon \mathrm{H}^{\bullet}(B) \xrightarrow{\sim} \mathrm{H}^{\bullet}(\pi_{\mathbb{P}}^{!}B).$$

The second part follows from Theorem 3.8.10 noting that all maps are compatible with the antipodal action. □

Theorem 3.5.4 and 3.5.5 together allow us to compute $\mathrm{H}^{\bullet}(\mathrm{Blup}(A, \iota^{!}A))$.

Corollary 3.5.6 *Let $\iota\colon N \hookrightarrow M$ be a closed transversal of a Lie algebroid $A \Rightarrow M$. Denote the blowdown map by $p_A\colon \mathrm{Blup}(A, \iota^{!}A) \to A$, the blowdown map of the base by $p\colon \mathrm{Blup}(M, N) \to M$, and the projection of the projective bundle by $\pi_{\mathbb{P}}\colon \mathbb{P} \to N$.*

1. *If* $\operatorname{codim} N$ *is odd,*

$$\mathrm{H}^{\bullet}(\mathrm{Blup}(A, \iota^{!}A)) = \mathrm{H}^{\bullet}(A) \oplus \mathrm{H}^{\bullet-1}(\iota^{!}A) \tag{3.53}$$

   and, under (3.53), $p_A^{*}\colon \mathrm{H}^{\bullet}(A) \xrightarrow{\sim} \mathrm{H}^{\bullet}(A) \oplus 0$.

2. *If* $\operatorname{codim} N$ *is even,* $\mathrm{H}^{\bullet}(\mathrm{Blup}(A, \iota^{!}A))$ *fits into a long exact sequence*

$$\cdots \to \mathrm{H}^{\bullet}(A) \xrightarrow{p_A^{*}} \mathrm{H}^{\bullet}(\mathrm{Blup}(A, \iota^{!}A)) \to$$
$$\xrightarrow{f} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A|_E) \oplus \mathrm{H}^{\bullet-1}(\pi_{\mathbb{P}}^{!}A) \xrightarrow{g} \mathrm{H}^{\bullet+1}(A) \to \dots$$

   where $\operatorname{im}(f) = X \oplus \mathrm{H}^{\bullet-1}(\pi_{\mathbb{P}}^{!}A)$ *for a subspace* $X \subseteq \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A|_E)$*, and* $g = i \circ \mathrm{pr}_{\mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A|_E)}$.

As a corollary, Theorem 3.5.4 also computes the de Rham cohomology of real projective blowups of manifolds.

Corollary 3.5.7 *Let $N \subseteq M$ be a closed and embedded submanifold.*

1. *If* $\operatorname{codim} N$ *is odd,*

$$p^{*}\colon \mathrm{H}^{\bullet}(M) \xrightarrow{\sim} \mathrm{H}^{\bullet}(\mathrm{Blup}(M, N)). \tag{3.54}$$

2. *If* $\operatorname{codim} N$ *is even, let $E \to N$ be a tubular neighbourhood of N in M. Then* $\mathrm{H}^{\bullet}(\mathrm{Blup}(M, N))$ *fits into a long exact sequence*

$$\cdots \to \mathrm{H}^{\bullet}(M) \xrightarrow{p^{*}} \mathrm{H}^{\bullet}(\mathrm{Blup}(M, N)) \xrightarrow{h} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(E) \xrightarrow{i} \mathrm{H}^{\bullet+1}(M) \to \dots \tag{3.55}$$

   Here, p_* first restricts a form to $\mathbb{P}$, then fibre-integrates and applies the Thom isomorphism.

Proof. The statement follows from Equation (3.49) for odd codimension and Equation (3.50) for even codimension of N, applied to the Lie algebroid $A = TM$, in which N is a transversal. □

3.5.1 The case of codimension 1

In this section, we prove Theorem 3.5.4, 1 for $\operatorname{codim}(N) = 1$. In this case, $p\colon \operatorname{Blup}(M,N) \xrightarrow{\simeq} M$ and $\mathbb{P} \simeq N$, hence the statement reduces to

$$\mathrm{H}^\bullet(\operatorname{Blup}(A, \iota^! A)) = \mathrm{H}^\bullet(A) \oplus \mathrm{H}^{\bullet-1}(\iota^! A). \tag{3.56}$$

We prove a slightly stronger statement, which can be seen as the Mazzeo-Melrose analogue for the blowup of a codimension 1 transversal, by adapting the proof in [MT14, Section 2.1] to our situation.

Theorem 3.5.8 (Mazzeo-Melrose for Lie algebroid transversals) *Let $\iota\colon N \hookrightarrow M$ be a closed codimension 1 transversal of a Lie algebroid $A \Rightarrow M$. Then the sequence*

$$0 \longrightarrow \Omega^\bullet(A) \xrightarrow{p_A^*} \Omega^\bullet(\operatorname{Blup}(A, \iota^! A)) \xrightarrow{j_{\operatorname{Blup}(a)}\big|_N} \Omega^{\bullet-1}(\iota^! A) \longrightarrow 0 \tag{3.57}$$

where j denotes the right insertion of sections and $a \in \Gamma(A)$ is Euler-like along N with $a\big|_N = 0$, is a split exact sequence of cochain complexes and does not depend on the choice of a. In particular, the Lie algebroid cohomology of the blowup is given by

$$\mathrm{H}^\bullet(\operatorname{Blup}(A, \iota^! A)) = \mathrm{H}^\bullet(A) \oplus \mathrm{H}^{\bullet-1}(\iota^! A). \tag{3.58}$$

We prove Theorem 3.5.8 in the remainder of this section. First, we collect properties of $\operatorname{Blup}(a) \in \Gamma(\operatorname{Blup}(A, \iota^! A))$ for $a \in \Gamma(A)$ as in the Theorem.

Lemma 3.5.9 *Let $\iota\colon N \hookrightarrow M$ be a closed Lie algebroid transversal of any codimension $\operatorname{codim} N \geq 1$. For an Euler-like section $a \in \Gamma(A)$ with $a\big|_N = 0$ (which exists by [BLM16, Lemma 3.9]), the blowup section $\operatorname{Blup}(a) \in \Gamma(\operatorname{Blup}(A, \iota^! A))$ satisfies the following:*

1. *$\sharp_{\operatorname{Blup}}(\operatorname{Blup}(a))$ is Euler-like along $\mathbb{P}$.*
2. *$\operatorname{Blup}(a) \in \Gamma(\operatorname{Blup}(A, \iota^! A))$ is nowhere vanishing on N and satisfies*
$$p_A(\operatorname{Blup}(a)\big|_{\mathbb{P}}) = 0, \tag{3.59}$$
where $p_A\colon \operatorname{Blup}(A, \iota^! A) \to A$ denotes the blowdown map.
3. *$\operatorname{Blup}(a)$ is Euler-like along the Lie subalgebroid $\operatorname{Blup}(A, \iota^! A)\big|_{\mathbb{P}}$.*
4. *If $a' \in \Gamma(A)$ is another section Euler-like along $\iota^! A$ with $a'\big|_N = 0$, we have*
$$(\operatorname{Blup}(a) - \operatorname{Blup}(a'))\big|_{\mathbb{P}} = 0. \tag{3.60}$$
In particular, (3.57) does not depend on the choice of a.

Proof. The first part follows from Corollary 3.3.6. For the second part, since $p_A(\operatorname{Blup}(a)\big|_{\mathbb{P}}) = 0$ is clear we, only have to show that $\operatorname{Blup}(a)\big|_{\mathbb{P}}$ is nowhere vanishing. For this, consider $\sharp\colon (A, \iota^! A) \to (TM, TN)$ and $\sharp(a)\colon (M, N) \to (TM, TN)$ as map of pairs. Then

$$\mathrm{Blup}(\sharp)(\mathrm{Blup}(a)) = \mathrm{Blup}(\sharp(a)) \in \Gamma(\mathrm{Blup}(TM, TN)). \tag{3.61}$$

But for the tubular neighbourhood, for which $\sharp(a)$ is the Euler vector field, it is easy to see that $\mathrm{Blup}(\sharp(a))$ is nowhere vanishing on $\mathbb{P}$ using Remark 3.3.8. Thus, $\mathrm{Blup}(a)$, too, must be nowhere vanishing on $\mathbb{P}$.

For the third part, let $f \in \mathscr{C}^\infty(\mathrm{Blup}(M, N))$ and $\mu \in \Gamma(\mathrm{Blup}(A, \iota^! A))$ be given. Then

$$[\mathrm{Blup}(a), f\mu]_{\mathrm{Blup}}\big|_{\mathbb{P}} = \underbrace{\sharp_{\mathrm{Blup}}(\mathrm{Blup}(a))}_{=0 \text{ on } \mathbb{P}}(f)\big|_{\mathbb{P}}\mu\big|_{\mathbb{P}} + f[\mathrm{Blup}(a), \mu]_{\mathrm{Blup}}\big|_{\mathbb{P}},$$

so it is enough to check the statement on $\mathrm{Blup}(\Gamma(A, \iota^! A))$, which generates $\Gamma(\mathrm{Blup}(A, \iota^! A))$. Next, we use $a \in \Gamma(A)$ to obtain an isomorphism of Lie algebroids

$$A\big|_E \cong \mathrm{pr}^! \iota^! A$$

for the tubular neighbourhood $\mathrm{pr}\colon E \to N$ corresponding to $\sharp(a)$. Since the statement is a local one around $\mathbb{P}$, it is enough to work in this neighbourhood. By [BLM16, Remark 3.19] this isomorphism maps $a \in \Gamma(A)$ to $(0, \xi_E)$, where ξ_E is the Euler vector field of E. Pick a linear connection on E with corresponding horizontal lift $\cdot^{\mathrm{hor}}$. Let $\{U_\alpha\}_\alpha$ be an open cover of N such that there are local frames $\{f_1, \dots, f_{\operatorname{rank} \iota^! A}\}$ of $(\iota^! A)\big|_U$ and $\{e_1, \dots, e_{\operatorname{codim} N}\}$ of $E\big|_U$. Then the collection $\{\tilde{f}_1, \dots, \tilde{f}_{\operatorname{rank} \iota^! A}, \tilde{e}_1, \dots, \tilde{e}_{\operatorname{codim} N}\}$ defined by

$$\begin{aligned} \tilde{f}_i &= (f_i \circ \mathrm{pr}, (\sharp_{\iota^! A}(f_i))^{\mathrm{hor}}) \\ \tilde{e}_j &= (0, e_j^{\mathrm{ver}}) \end{aligned}$$

yields a local frame of $\mathrm{pr}^! \iota^! A\big|_{E_U}$. The module $\Gamma(\mathrm{pr}^! \iota^! A\big|_{E_U}, \iota^! A\big|_U)$ is generated by $\tilde{f}_i$'s and $g\tilde{e}_j$'s, where $g \in \mathcal{I}_N$ is in the vanishing ideal of N. By definition we have

$$[\mathrm{Blup}(a), \mathrm{Blup}(b)]_{\mathrm{Blup}} = \mathrm{Blup}([a, b])$$

and want to show that

$$\mathrm{Blup}([a, b])\big|_{\mathbb{P}} = 0$$

for any such section b. For this, note that it is enough to show that $[a, b]$ vanishes to second order along N. Clearly, we have

$$[a, \tilde{f}_i] = [(0, \xi_E), (f_i \circ \mathrm{pr}, \sharp_{\iota^! A}(f_i)^{\mathrm{hor}})] = (0, [\xi_E, \sharp_{\iota^! A}(f_i)^{\mathrm{hor}}]) = 0.$$

Thus, let $g \in \mathcal{I}_N$ be given. Then

$$\begin{aligned} [a, g\tilde{e}_j] &= (0, \xi_E(g) e_j^{\mathrm{ver}}) + g(0, [\xi_E, e_j^{\mathrm{ver}}]) \\ &= (0, \xi_E(g) e_j^{\mathrm{ver}} - g e_j^{\mathrm{ver}}) \\ &= 0 \mod \mathcal{I}_N^2 \end{aligned}$$

as $\xi_E(g) = g \mod \mathcal{I}_N^2$.

For the last part, let $a' \in \Gamma(A)$ be another choice of Euler-like section. In the trivialisation induced by a we can write

$$a' = \sum_i g_i \tilde{f}_i + \sum_j h_j \tilde{e}_j$$

for $g_i, h_j \in \mathscr{C}^\infty(M)$. Since by assumption $a'\big|_N = 0$, $g_i \in \mathcal{I}_N$ follows. Since $\sharp_A(a')$ is Euler-like along N,

$$\xi_E - \sum_j h_j \tilde{e}_j \in \mathcal{I}_N^2 \Gamma(TM)$$

vanishes to second order. In conclusion, $\mathrm{Blup}(a - a')\big|_{\mathbb{P}} = 0$. ☐

Next, we show that the insertion of $\mathrm{Blup}(a)\big|_N$ is a cochain map. Clearly the restriction to N itself is a cochain map, so the crucial part happens inside the Lie subalgebroid $\mathrm{Blup}(A, \iota^! A)\big|_N$. We first note that, in the case of $\operatorname{codim} N = 1$, the Lie algebroid $\mathrm{Blup}(A, \iota^! A)\big|_N$ is an abelian extension according to the definition in [Mac05].

Lemma 3.5.10 *Let $\iota\colon N \hookrightarrow M$ be a closed transversal of a Lie algebroid $A \Rightarrow M$ of any codimension. Then* $\mathrm{Blup}(A, \iota^! A)\big|_{\mathbb{P}}$ *fits into a short exact sequence of Lie algebroids*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, Rightarrow] & \mathrm{Blup}(A, \iota^! A)\big|_{\mathbb{P}} \arrow[r, "p_A"] \arrow[d, Rightarrow] & \iota^! A \arrow[r] \arrow[d, Rightarrow] & 0 \\ & \mathbb{P} \arrow[r, "\mathrm{id}_{\mathbb{P}}"] & \mathbb{P} \arrow[r, "p"] & N & \end{tikzcd} \tag{3.62}$$

where $L := \ker p_A\big|_{\mathbb{P}}$ and $i\colon L \hookrightarrow$ $\mathrm{Blup}(A, \iota^! A)\big|_{\mathbb{P}}$ *denotes the inclusion.*

Moreover, if $\operatorname{codim} N = 1$, *the sequence* (3.62) *is an abelian extension of Lie algebroids.*

Proof. Since $p_A\big|_{\mathbb{P}}\colon \mathrm{Blup}(A, \iota^! A)\big|_{\mathbb{P}} \to A\big|_N$ is a vector bundle morphism with image given by $\iota^! A$, it follows that $L = \ker p_A\big|_{\mathbb{P}}$ is a vector bundle of rank $\operatorname{codim} N$ and (3.62) is a short exact sequence of Lie algebroids. To show that, in case of $\operatorname{codim} N = 1$, the Lie algebroid L is abelian, note that the anchor of the blowup makes the diagram

$$\begin{tikzcd} \mathrm{Blup}(A, \iota^! A)\big|_N \arrow[r, "p_A"] \arrow[d, "\sharp_{\mathrm{Blup}}"'] & A\big|_N \arrow[d, "\sharp"] \\ TM\big|_N \arrow[r, "="] & TM\big|_N \end{tikzcd}$$

commute, from which $L \subseteq \ker \sharp_{\mathrm{Blup}}$ follows. Thus, the inherited anchor is zero, turning L into an abelian Lie algebroid of rank 1. ☐

Remark 3.5.11 For an arbitrary codimension of N, the extension (3.62) is not abelian. Consider $A = T\mathbb{R}^2 \Rightarrow \mathbb{R}^2$ and $\iota\colon N = \{0\} \hookrightarrow \mathbb{R}^2$. Then $L = \operatorname{Blup}(A, \iota^! A)\big|_{\mathbb{P}}$, and

$$[\operatorname{Blup}(x\partial_x), \operatorname{Blup}(x\partial_y)]_{\mathrm{Blup}} = \operatorname{Blup}(x\partial_x)$$

does not vanish on $\mathbb{P}$.

By Lemma 3.5.10, for $\iota\colon N \hookrightarrow M$ a transversal of a Lie algebroid $A \Rightarrow M$ of codimension 1 there is a representation of $\iota^! A$ on L given by

$$\nabla_b \eta = [\tilde{b}, \eta] = [\hat{b}, \hat{\eta}]\big|_N \tag{3.63}$$

for $\tilde{b} \in \Gamma(\operatorname{Blup}(A, \iota^! A)\big|_N)$ with $p(\tilde{b}) = b$ or, alternatively, $\hat{b}, \hat{\eta}$ extensions of $\tilde{b}, \eta$ to a section of $\operatorname{Blup}(A, \iota^! A)$ [Mac05, Proposition 3.3.20].

Key to showing that $j_{\operatorname{Blup}(a)}\big|_N$ in (3.57) is a chain map is to see that it is enough for $\operatorname{Blup}(a)\big|_N$ to be constant with respect to the representation (3.63) of $\iota^! A$.

Lemma 3.5.12 *Let $A \Rightarrow M$ be a Lie algebroid, $\iota\colon N \hookrightarrow M$ a closed codimension 1 transversal and $L = \ker p_A\big|_N$. Suppose $\eta \in \Gamma(L)$ is constant with respect to the action of $\iota^! A$, i.e.*

$$\nabla_b \eta = 0 \tag{3.64}$$

for all $b \in \Gamma(\iota^! A)$. Then the map

$$\Omega^\bullet(\operatorname{Blup}(A, \iota^! A)) \ni \omega \mapsto j_\eta \omega\big|_N \in \Omega^{\bullet-1}(\iota^! A) \tag{3.65}$$

is a chain map, where for $\omega \in \Omega^k(\operatorname{Blup}(A, \iota^! A))$ and $b_1, \dots, b_{k-1} \in \Gamma(\iota^! A)$ we define

$$j_\eta \omega\big|_N(b_1, \dots, b_{k-1}) = \omega\big|_N(\tilde{b}_1, \dots, \tilde{b}_{k-1}, \eta). \tag{3.66}$$

Here, $\tilde{b}_i$ is any section of $\Gamma(\operatorname{Blup}(A, \iota^! A)\big|_N)$ such that $p_A(\tilde{b}_i) = b_i$.

Proof. Let $\omega \in \Omega^k(\operatorname{Blup}(A, \iota^! A))$ and $b_0, \dots, b_{k-1} \in \Gamma(\iota^! A)$ be given with corresponding $\tilde{b}_i$. Then we get

$$\begin{aligned}
j_\eta \mathrm{d}\omega\big|_N(b_0, \dots, b_{k-1}) &= (\mathrm{d}\omega)\big|_N(b_0, \dots, b_{k-1}, \eta) \\
&= \sum_{i=0}^{k-1} (-1)^i \sharp_{\mathrm{Blup}}(\tilde{b}_i)\omega\big|_N(\tilde{b}_0, \dots, \overset{i}{\wedge}, \dots, \tilde{b}_{k-1}, \eta) \\
&\quad + (-1)^k \sharp_{\mathrm{Blup}}(\eta)\omega\big|_N(\tilde{b}_0, \dots, \tilde{b}_{k-1}) \\
&\quad + \sum_{0 \le i < j \le k-1} (-1)^{i+j} \omega\big|_N([\tilde{b}_i, \tilde{b}_j], \tilde{b}_0, \dots, \overset{i}{\wedge}, \dots, \overset{j}{\wedge}, \dots, \tilde{b}_{k-1}, \eta) \\
&\quad + (-1)^k \sum_{i=0}^{k-1} (-1)^i \omega\big|_N([\tilde{b}_i, \eta], \tilde{b}_0, \dots, \overset{i}{\wedge}, \dots, \tilde{b}_{k-1}) \\
&= \mathrm{d}_{\iota^! A}(j_\eta \omega\big|_N)(b_0, \dots, b_{k-1}) + (-1)^k \sharp_{\mathrm{Blup}}(\eta)\omega\big|_N(\tilde{b}_0, \dots, \tilde{b}_{k-1}) \\
&\quad + (-1)^k \sum_{i=0}^{k-1} (-1)^i \omega\big|_N([\tilde{b}_i, \eta], \tilde{b}_0, \dots, \overset{i}{\wedge}, \dots, \tilde{b}_{k-1}).
\end{aligned}$$

Thus, (3.65) defines a chain map if and only if the expression

$$\sharp_{\mathrm{Blup}}(\eta)\omega\big|_N(\tilde{b}_0,\dots,\tilde{b}_{k-1})+\sum_{i=0}^{k-1}(-1)^i\omega\big|_N([\tilde{b}_i,\eta],\tilde{b}_0,\dots,\overset{i}{\wedge},\dots,\tilde{b}_{k-1})$$

is zero. But under the assumptions on η every single summand vanishes. ☐

Corollary 3.5.13 *Let $\iota\colon N\hookrightarrow M$ be a closed transversal of a Lie algebroid $A\Rightarrow M$ with* $\operatorname{codim} N=1$ *and $a\in\Gamma(A)$ the Euler-like section from Lemma 3.5.9. Then the map*

$$j_{\mathrm{Blup}(a)}\big|_N\colon \Omega^\bullet(\mathrm{Blup}(A,\iota^!A))\to\Omega^{\bullet-1}(\iota^!A) \tag{3.67}$$

is a chain map.

As a last ingredient we need the concept of an adapted distance function, see [MT14, Section 2.1].

Definition 3.5.14 Let $E\to N$ be a vector bundle equipped with a metric. A function $\lambda\colon E\setminus N\to\mathbb{R}_0^+$ is called an adapted distance function if the following hold:

1. $\lambda\in\mathscr{C}^\infty(E\setminus N)$.
2. For all $x\in E\setminus N$ with $|x|<\frac{1}{2}$, $\lambda(x)=|x|$.
3. $\lambda(x)=1$ for all $x\in E$ with $|x|>1$.

Note that such a function always exists, see e.g. [Geu17, Section 8.5]. With the notion of an adapted distance function at hand, we can proof Theorem 3.5.8.

Proof of Theorem 3.5.8. First note that $j_{\mathrm{Blup}(a)}\big|_N\circ p_A^*=0$ is clear as $a\big|_N=0$. Suppose that $\omega\in\Omega^k(\mathrm{Blup}(A,\iota^!A))$ is mapped to 0 under $j_{\mathrm{Blup}(a)}\big|_N$. Since we have the short exact sequence

$$0\longrightarrow L\longrightarrow \mathrm{Blup}(A,\iota^!A)_N\xrightarrow{p_A}\iota^!A\longrightarrow 0$$

this just means that $\omega\big|_N$ actually is the pullback of a form on $\iota^!A$. By computing in local coordinates one can then easily show that $\omega\in p^*\Omega^k(A)$: If $\{b_1,\dots,b_k,e\}$ is a local frame over an adapted chart $U\subseteq M$, $N\cap U=\{x=0\}$, such that the collection of $b's$ yield a local frame for $\iota^!A$ when restricted to N, a form of the blowup is generated by forms that locally looks like

$$\omega=f\frac{e^*}{x}\wedge(b^*)^I+(b^*)^J,$$

for some $f\in\mathscr{C}^\infty(N)$, denoting the dual frames with a $\cdot^*$ and I,J multi-indices, meaning that e.g. $(b^*)^{\{i_1,i_2\}}=b_{i_1}^*\wedge b_{i_2}^*$. Then $j_{\mathrm{Blup}(a)}\omega\big|_N=0$ implies that $f=xg$, since $\mathrm{Blup}(a)\big|_N$ is nowhere vanishing. Finally, by Corollary 3.5.13 the sequence (3.57) is indeed a sequence of chain complexes.

To define a splitting, let $\lambda(r)$ be an adapted distance function on E. Then $\mathrm{d}\log\lambda$ is a form on $E\setminus N$ dual to the vertical Euler vector field is a neighbourhood of N, thus, there exists $\sharp^*\mathrm{d}\log\lambda \in \Omega^1(A\big|_{E\setminus N})$. The corresponding form $\alpha = \sharp^*_{\mathrm{Blup}}\mathrm{d}\log\lambda$ extends smoothly to N (as can again be seen in local coordinates) and satisfies $\alpha(\mathrm{Blup}(a)) = 1$ on N by continuity. Moreover, it is compactly supported in fibre direction of E (by definition of λ) and closed, as $\sharp^*_{\mathrm{Blup}}$ is a chain map and $\mathrm{d}\log\lambda$ is closed on the dense subset $E\setminus N$. Thus, we can define the map

$$\Omega^{\bullet-1}(\iota^!A) \ni \tilde{\omega} \mapsto p_A^*(\mathrm{pr}^!)^*\tilde{\omega}\wedge\alpha \in \Omega^\bullet(\mathrm{Blup}(A,\iota^!A)).$$

By closedness of α this is a chain map, thus defining the desired splitting. In particular, $j_{\mathrm{Blup}(a)}\big|_N$ is surjective, hence the sequence is exact. But now it is clear that in the decomposition

$$\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)) = \mathrm{H}^\bullet(A)\oplus\mathrm{H}^{\bullet-1}(\iota^!A),$$

the pullback by the blowdown map maps into the first factor. □

Remark 3.5.15 One way to interpret the closed 1-form $\alpha = \sharp^*_{\mathrm{Blup}}\mathrm{d}\log\lambda$ from the proof of Theorem 3.5.8 is as a Lie algebroid splitting for (3.62): Restricted to N it is still closed, hence we can identify

$$\iota^!A \simeq \ker\alpha\big|_N \subseteq \mathrm{Blup}(A,\iota^!A)\big|_N \tag{3.68}$$

as Lie algebroids. Thus, by Corollary 2.4.2 we find

$$\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)\big|_N) = \mathrm{H}^\bullet(\iota^!A)\oplus\mathrm{H}^{\bullet-1}(\iota^!A), \tag{3.69}$$

using that L is trivialisable such that the representation of $\iota^!A$ becomes the trivial one. This observation alone is enough to prove Theorem 3.5.8. Indeed, let $E\to N$ denote the tubular neighbourhood induced by the chosen Euler-like section $a\in\Gamma(A)$ and $\mathbb{L}$ its tautological line bundle (see Remark 3.3.5). Then there is an isomorphism

$$\mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)\big|_{\mathbb{L}}) \cong \mathrm{H}^\bullet(\mathrm{Blup}(A,\iota^!A)\big|_N) \tag{3.70}$$

induced by the restriction, see [MS24, Theorem 3.27] or Theorem 2.2.27 in combination with Lemma 3.5.9. Together with $\mathrm{H}^\bullet(A\big|_E)\cong\mathrm{H}^\bullet(\iota^!A)$ we find that in cohomology we obtain a long exact sequence

$$\ldots\to\mathrm{H}^k(A\big|_E)\xrightarrow{p_A^*}\mathrm{H}^k(\mathrm{Blup}(A\big|_E,\iota^!A))\xrightarrow{f}\mathrm{H}^{k-1}(\iota^!A)\xrightarrow{g}\mathrm{H}^{k+1}(A\big|_E)\to\ldots \tag{3.71}$$

in which f is surjective and g is zero. Moreover, this identifies

$$\mathrm{H}^\bullet\left(\frac{\Omega^\bullet(\mathrm{Blup}(A\big|_E,\iota^!A))}{p_A^*\Omega^\bullet(A\big|_E)}\right) = \mathrm{H}^{\bullet-1}(\iota^!A). \tag{3.72}$$

Then Theorem 3.5.8 can be proven with the same local-to-global technique used to prove Theorem 3.5.4, 3 in Section 3.5.3.

3.5.2 From arbitrary codimension to codimension 1

The next step to proving Theorem 3.5.4 is to show that for a closed transversal $\iota\colon N \hookrightarrow M$ of any codimension the blowup is isomorphic to the blowup of a codimension 1 transversal in a different Lie algebroid, namely in $p^!A$. Since N is a transversal and, for every $x \in \mathbb{P}$, $T_{p(x)}N \subseteq T_x p T_x \mathrm{Blup}(M,N)$, the blowdown map is transverse to the anchor.Hence, $p^!A$ is a well-defined Lie algebroid. Moreover, at the end of the subsection we show that $(p^!)^*$, like p_A^*, is an isomorphism when restricted to flat forms.

Proposition 3.5.16 *Let $A \Rightarrow M$ be a Lie algebroid and $\iota\colon N \hookrightarrow M$ a closed transversal. Denoting the blowdown map of the base by $p\colon \mathrm{Blup}(M,N) \to M$ and the inclusion of the projective bundle by $\iota_{\mathbb{P}}\colon \mathbb{P} \hookrightarrow \mathrm{Blup}(M,N)$, there is an isomorphism of Lie algebroids*

$$\mathrm{Blup}(A, \iota^!A) \cong \mathrm{Blup}(p^!A, \iota_{\mathbb{P}}^! p^!A) \tag{3.73}$$

over the identity. The corresponding map on sections is given by

$$\Gamma(\mathrm{Blup}(A, \iota^!A)) \ni b \mapsto (p_A(b), \sharp_{\mathrm{Blup}}(b)) \in \Gamma(p^!A, \iota_{\mathbb{P}}^! p^!A), \tag{3.74}$$

identifying $\Gamma(\mathrm{Blup}(p^!A, \iota_{\mathbb{P}}^! p^!A)) = \Gamma(p^!A, \iota_{\mathbb{P}}^! p^!A)$. Here, $p_A\colon \mathrm{Blup}(A, \iota^!A) \to A$ denotes the blowdown map. In particular, under this isomorphism p_A and $p^! \circ p_{p^!A}$ coincide.

Proof. First note that $\iota_{\mathbb{P}}\colon \mathbb{P} \hookrightarrow \mathrm{Blup}(M,N)$ is a transversal in $p^!A$ because $\iota\colon N \hookrightarrow M$ is. For $X \in \Gamma(TM, TN)$ we denote by $\tilde{X} \in \Gamma(T\mathrm{Blup}(M,N))$ the vector field with $\tilde{X} \sim_p X$, which exists by Lemma 3.3.4. We first check that (3.74) is well-defined. Since both spaces are $\mathscr{C}^\infty(\mathrm{Blup}(M,N))$-modules and (3.74) is compatible with the module structure, it is enough to check well-definedness on a set of generators, namely $\mathrm{Blup}(\Gamma(A, \iota^!A))$. Thus, let $a \in \Gamma(A, \iota^!A)$ be given. Then

$$\mathrm{Blup}(a) \mapsto (p_A(\mathrm{Blup}(a)), \sharp_{\mathrm{Blup}}(\mathrm{Blup}(a))) = (a \circ p, \widetilde{\sharp(a)}).$$

Since $\widetilde{\sharp(a)} \sim_p \sharp(a)$, $(a \circ p, \widetilde{\sharp(a)}) \in \Gamma(p^!A)$ follows. Moreover, by Lemma 3.3.4, $\widetilde{\sharp(a)}$ is tangent to $T\mathbb{P}$, so $\mathrm{Blup}(a)$ maps to a section in $\Gamma(p^!A, \iota_{\mathbb{P}}^! p^!A)$ as claimed.

Now it is immediate to see that outside of $\mathbb{P}$, Equation (3.74) gives an isomorphism (of vector bundles). Thus, it is enough to consider the neighbourhood around N provided by the normal form theorem 3.5.1. Suppose $\mathrm{pr}\colon E = M \to N$ is a vector bundle and $A = \mathrm{pr}^! \iota^! A$. As in the proof of Lemma 3.5.9, let $\{a_1, \dots, a_k\}$ be a local frame of $\iota^!A$ and $\{s_1, \dots, s_{\operatorname{codim} N}\}$ a local frame of E over some common open $U \subseteq N$ (we will use the dual frame on E as fibre coordinates on $E\big|_U$). According to Remark 3.3.8, for $\beta \in \{1, \dots, \operatorname{codim} N\}$ we obtain a frame for $\mathrm{Blup}(A, \iota^!A)\big|_{U_\beta}$ by

$$\{\mathrm{Blup}(\tilde{a}_1), \dots, \mathrm{Blup}(\tilde{a}_k), \mathrm{Blup}(s^\beta \tilde{s}_1), \dots, \mathrm{Blup}(s^\beta \tilde{s}_{\operatorname{codim} N})\},$$

where

$$\begin{aligned} \tilde{a}_i &= (a_i \circ \mathrm{pr}, (\sharp_{\iota^!A}(a_i))^{\mathrm{hor}}) \\ \tilde{s}_j &= (0, s_j^{\mathrm{ver}}) \end{aligned}$$

for the horizontal lift of some linear connection on $E \to N$. Then (3.74) maps

$$\begin{aligned} \operatorname{Blup}(\tilde{a}_j) &\mapsto (\tilde{a}_j \circ p, \widetilde{\sharp(a_j)}), \\ \operatorname{Blup}(s^\beta \tilde{s}_\alpha) &\mapsto (0, \widetilde{s^\beta \frac{\partial}{\partial s^\alpha}}). \end{aligned}$$

The above collection of sections still span $\Gamma(p^! A, \iota_{\mathbb{P}}^! p^! A)$ locally over U_β as

$$\widetilde{s^\beta \frac{\partial}{\partial s^\alpha}} = \begin{cases} \frac{\partial}{\partial \tilde{s}^\alpha} & \text{if } \alpha \neq \beta \\ s^\beta \frac{\partial}{\partial \tilde{s}^\beta} - \sum_{\gamma \neq \alpha} \tilde{s}^\gamma \frac{\partial}{\partial \tilde{s}^\gamma} & \text{if } \alpha = \beta \end{cases}$$

by Lemma 3.3.4. Finally, we check that it is a morphism of Lie algebroids. Clearly, (3.74) preserves anchors by its very definition. Thus, it is enough to check compatibility with the Lie bracket using a set of generators. Let $a, a' \in \Gamma(A, \iota^! A)$ be given. Then

$$\begin{aligned} [\operatorname{Blup}(a), \operatorname{Blup}(a')] \mapsto &(p_A(\operatorname{Blup}([a, a'])), \widetilde{\sharp([a, a'])}) \\ = &([a, a'] \circ p, \widetilde{[\sharp(a), \sharp(a')]}) \\ = &([a, a'] \circ p, [\widetilde{\sharp(a)}, \widetilde{\sharp(a')}]) \\ = &[(a \circ p, \widetilde{\sharp(a)}), (a' \circ p, \widetilde{\sharp(a')})], \end{aligned}$$

hence (3.74) constitutes an isomorphism of Lie algebroids. □

Using Theorem 3.5.8, the identification $\operatorname{Blup}(A, \iota^! A) \cong \operatorname{Blup}(p^! A, \iota_{\mathbb{P}}^! p^! A)$ from Proposition 3.5.16 implies the following.

Corollary 3.5.17 *Let $\iota\colon N \hookrightarrow M$ be a transversal of a Lie algebroid $A \Rightarrow M$. Then*

$$\mathrm{H}^\bullet(\operatorname{Blup}(A, \iota^! A)) = \mathrm{H}^\bullet(p^! A) \oplus \mathrm{H}^{\bullet - 1}(\iota_{\mathbb{P}}^! p^! A), \tag{3.75}$$

where $p\colon \operatorname{Blup}(M, N) \to M$ denotes the blowdown map of the base manifolds and $\iota_{\mathbb{P}}\colon \mathbb{P} \hookrightarrow \operatorname{Blup}(M, N)$ the inclusion of the projective bundle. Under this identification, $p_A^*\colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(\operatorname{Blup}(A, \iota^! A))$ maps into $\mathrm{H}^\bullet(p^! A)$ and is given by $(p^!)^*$.

Thus, to compute the cohomology of the blowup one needs to compute the cohomology of $p^! A$. By the normal form theorem 3.5.1 and (3.44), locally this comes down to comparing the cohomology of $\iota^! A$ to that of $\pi_{\mathbb{P}}^! A$, the pullback to a projective bundle. And since in this case, $(p^!)^*$ too constitutes an isomorphism when restricted to flat forms, the local picture will be enough.

Lemma 3.5.18 *Let $N \subseteq M$ be a closed transversal of a Lie algebroid $A \Rightarrow M$. Denoting the blowdown map of the base by $p\colon \operatorname{Blup}(M, N) \to M$, the map*

$$(p^!)^*\colon \Omega_N^\bullet(A) \to \Omega_{\mathbb{P}}^\bullet(p^! A) \tag{3.76}$$

is an isomorphism.

Proof. We know that the diagram

$$\begin{tikzcd}
\Omega^\bullet(\mathrm{Blup}(A, \iota^! A)) \arrow[r, "\Phi^*", "\simeq"'] & \Omega^\bullet(\mathrm{Blup}(p^! A, p^! \iota^! A)) \\
\Omega^\bullet(A) \arrow[u, "p_A^*"] \arrow[r, "(p^!)^*"] & \Omega^\bullet(p^! A) \arrow[u, "p_{p^! A}^*"']
\end{tikzcd}$$

commutes, see Proposition 3.5.16 for the upper isomorphism. We see that the maps factor to flat forms, and there p_A^* and $p_{p^! A}^*$ are isomorphisms, thus so is $(p^!)^*$. ☐

Lemma 3.5.19 *Let $E \to N$ be the tubular neighbourhood corresponding to an Euler-like section $a \in \Gamma(A)$. Then the inclusions $\iota^! A \hookrightarrow A\big|_E$ and $\pi_{\mathbb{P}}^! \iota^! A \hookrightarrow p^! A\big|_E$ induce isomorphisms such that*

$$\begin{tikzcd}
\mathrm{H}^\bullet(A\big|_E) \arrow[r, phantom, "\cong"] \arrow[d, "(p^!)^*"'] & \mathrm{H}^\bullet(\iota^! A) \arrow[d, "(\pi_{\mathbb{P}}^!)^*"] \\
\mathrm{H}^\bullet(p^! A\big|_E) \arrow[r, phantom, "\cong"] & \mathrm{H}^\bullet(\pi_{\mathbb{P}}^! \iota^! A)
\end{tikzcd} \tag{3.77}$$

commutes. Here, $\pi_{\mathbb{P}} \colon \mathbb{P} \to N$ denotes the projection of the projective bundle.

If codim N *is odd, then all maps in* (3.77) *are isomorphisms.*

Proof. The inclusions yield a commutative diagram

$$\begin{tikzcd}
A\big|_E & \iota^! A \arrow[l] \\
p^! A\big|_E \arrow[u, "p^!"] & \pi_{\mathbb{P}}^! \iota^! A \arrow[l] \arrow[u, "\pi_{\mathbb{P}}^!"']
\end{tikzcd}$$

of Lie algebroids, leading to (3.77) in cohomology, where the inclusions become isomorphisms by (3.44). If codim N is odd, Theorem 3.5.5 shows the remaining statement. ☐

3.5.3 Proof of Theorem 3.5.4

The third part of Theorem 3.5.4 is more technical, thus we prove the remaining statements of Theorem 3.5.4 first.

Proof of Theorem 3.5.4, 1 and 2. For the first part, the isomorphism

$$\mathrm{H}^\bullet(\mathrm{Blup}(A, \iota^! A)) \cong \mathrm{H}^\bullet(\mathrm{Blup}(p^! A, \pi_{\mathbb{P}}^! A))$$

is Proposition 3.5.16, while $\mathrm{H}^\bullet(\mathrm{Blup}(A, \iota^! A)) \cong \mathrm{H}^\bullet(p^! A) \oplus \mathrm{H}^{\bullet-1}(\pi_{\mathbb{P}}^! A)$ is Theorem 3.5.8 since $\mathbb{P}$ is a transversal of codimension 1 inside $p^! A$. For the second part note that we find a neighbourhood E of $\mathbb{P}$ in $\operatorname{Blup}(M,N)$ such that $\mathrm{H}^\bullet(p^! A|_E) \cong \mathrm{H}^\bullet(\pi_{\mathbb{P}}^! A)$ by means of an Euler-like section. Thus, we have

$$\mathrm{H}^\bullet\left(\frac{\Omega^\bullet(p^! A)}{(p^!)^*\Omega^\bullet(A)}\right) \cong \mathrm{H}^\bullet\left(\frac{\Omega^\bullet(p^! A|_E)}{(p^!)^*\Omega^\bullet(A|_E)}\right) = 0,$$

where the first isomorphism is a direct consequence of Lemma 3.5.18 (see also Theorem 3.4.1), and the second identification follows from Lemma 3.5.19. From this,

$$(p^!)^* \colon \mathrm{H}^\bullet(A) \xrightarrow{\simeq} \mathrm{H}^\bullet(p^! A)$$

follows at once, implying $\mathrm{H}^\bullet(\operatorname{Blup}(A, \iota^! A)) = \mathrm{H}^\bullet(A) \oplus \mathrm{H}^{\bullet-1}(\iota^! A)$ by the first part. $\square$

To show Theorem 3.5.4, 3, we first work in a tubular neighbourhood $\mathrm{pr}\colon E \to N$ induced by an Euler-like section of A along N.

Lemma 3.5.20 *Let* $\mathrm{pr}\colon E \to N$ *be a vector bundle with zero section* $\iota\colon N \hookrightarrow E$ *and* $B \Rightarrow N$ *a Lie algebroid. Then there is a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^\bullet(\mathrm{pr}^! B) \xrightarrow{(p^!)^*} \mathrm{H}^\bullet(p^! \mathrm{pr}^! B) \xrightarrow{(p^!)_*} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(\mathrm{pr}^! B) \xrightarrow{i} \mathrm{H}^{\bullet+1}(\mathrm{pr}^! B) \longrightarrow \ldots \tag{3.78}$$

Here, $p\colon \operatorname{Blup}(E,N) \to E$ *is the blowdown map of the base,* $(p^!)_*$ *denotes the composition of the Thom isomorphism after fibre integrating the restriction of a form to* $\mathbb{P}$*, and* $i\colon \mathrm{H}^\bullet_{\mathrm{cv}}(\mathrm{pr}^! B) \to \mathrm{H}^\bullet(\mathrm{pr}^! B)$ *denotes the canonical map that regards a form compactly supported in fibre direction as just a form on* $\mathrm{pr}^! B$*.*

Proof. This follows from Theorem 3.5.5, 2 using the same reasoning as in Lemma 3.8.11, as we again have both

$$\mathrm{H}^\bullet(\mathrm{pr}^! B) \cong \mathrm{H}^\bullet(B) \text{ and}$$
$$\mathrm{H}^\bullet(p^! B) \cong \mathrm{H}^\bullet(\pi_{\mathbb{P}}^! B)$$

by means of the respective restrictions. $\square$

This already gives Theorem 3.5.4, 3 in the local setting. To prove the global statement, we use that by Lemma 3.5.18 for any tubular neighbourhood $E \to N$,

$$Q^\bullet := \mathrm{H}^\bullet\left(\frac{\Omega^\bullet(p^! A)}{(p^!)^*\Omega^\bullet(A)}\right) = \mathrm{H}^\bullet\left(\frac{\Omega^\bullet(p^! A|_E)}{(p^!)^*\Omega^\bullet(A|_E)}\right) =: Q^\bullet_E, \tag{3.79}$$

where equality already holds on the level of chain complexes. The plan is to identify $Q^\bullet$ and $\mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A|_E)$ to complete the proof of Theorem 3.5.4. We can define a map $Q^\bullet_E \to \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A|_E)$ in the following way: First, pick any splitting σ of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Omega^\bullet(A|_E) \arrow[r, "(p^!)^*"] & \Omega^\bullet(p^! A|_E) \arrow[r, shift left, "\tau"] & \frac{\Omega^\bullet(p^! A|_E)}{(p^!)^*\Omega^\bullet(A|_E)} \arrow[l, shift left, dashed, "\sigma"] \arrow[r] & 0 \end{tikzcd}$$

and let $\chi \in \mathscr{C}^\infty(E)$ be a smooth function with compact vertical support, such that $\chi = 1$ in a neighbourhood of N. Then also $\tilde{\sigma} = p^*\chi\sigma$ defines a splitting for τ, since by Lemma 3.5.18 τ depends only on local data around $\mathbb{P}$, and there the map did not change. We proceed now to define a map the same way one would construct the edge endomorphism in the corresponding long exact sequence in cohomology: For $[\lambda] \in Q_E^\bullet$ we know that

$$\tau \mathrm{d}_{p^! A\big|_E} \tilde{\sigma}\lambda = \mathrm{d}\tau\tilde{\sigma}\lambda = \mathrm{d}\lambda = 0,$$

so we can find a unique $\eta \in \Omega^\bullet(A\big|_E)$ such that $(p^!)^*\eta = \mathrm{d}_{p^! A\big|_E} \tilde{\sigma}\lambda$. Consequently, η has compact vertical support, and since $(p^!)^*$ is injective, $\mathrm{d}_{A\big|_E} \eta = 0$. We map $[\lambda]$ to $[\eta] \in \mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A\big|_E)$. This is well-defined: If $\lambda' \in [\lambda]$ with corresponding η', then there exists λ_* in the quotient complex such that $\lambda' - \lambda = \mathrm{d}\lambda_*$. Then there exists a unique $\eta_* \in \Omega^\bullet(A\big|_E)$ that satisfies

$$(p^!_A)^*\eta_* = \tilde{\sigma}\lambda' - \tilde{\sigma}\lambda - \mathrm{d}_{p^! A\big|_E} \tilde{\sigma}\lambda_*.$$

Then η_* has compact vertical support and $\mathrm{d}_{A\big|_E} \eta_* = \eta' - \eta$, showing that the cohomology class $[\eta] \in \mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A\big|_E)$ does not depend on the chosen representative.

Lemma 3.5.21 *The above constructed map $\Psi \colon Q_E^\bullet \to \mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A_E)$ is an isomorphism.*

Proof. We will show this by using the 5-lemma [Mac63, Chapter 1, Lemma 3.3]. Note that $Q_E^\bullet$ and $\mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A_E)$ fit into a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^\bullet(A\big|_E) \arrow[r, "(p^!)^*"] \arrow[d] & \mathrm{H}^\bullet(p^! A\big|_E) \arrow[r, "\tau"] \arrow[d] & Q_E^\bullet \arrow[r, "\delta"] \arrow[d, "\Psi"'] & {} \\
\cdots \arrow[r] & \mathrm{H}^\bullet(A\big|_E) \arrow[r, "(p^!)^*"] & \mathrm{H}^\bullet(p^! A\big|_E) \arrow[r, "-2(p^!)_*"] & \mathrm{H}_{\mathrm{cv}}^{\bullet+1}(A\big|_E) \arrow[r, "i"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & \mathrm{H}^{\bullet+1}(A\big|_E) \arrow[r, "(p^!)^*"] \arrow[d] & \mathrm{H}^{\bullet+1}(p^! A\big|_E) \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r, "i"] & \mathrm{H}^{\bullet+1}(A\big|_E) \arrow[r, "(p^!)^*"] & \mathrm{H}^{\bullet+1}(p^! A\big|_E) \arrow[r] & \cdots
\end{tikzcd}$$

where δ denotes the edge homomorphism and unlabeled vertical arrows are the identity. By the 5-Lemma, if the diagram commutes, Ψ is an isomorphism. Thus, we have to check if the squares left and right to Ψ are commuting. For the square on the right this is obvious, since Ψ is constructed as the edge homomorphism.

Consider the square on the left and let $[\omega] \in \mathrm{H}^\bullet(p^! A\big|_E)$ be given. We first compute $-2(p^!)_*[\omega]$ explicitly and then check that it coincides with $\Psi\tau[\omega]$. By the choice of E, we find $\eta \in \Omega^\bullet(\pi^!_{\mathbb{P}}\iota^! A)$ such that $[\omega] = [(\pi^!_{\mathbb{L}})^*\eta]$ by (3.44). Here, $\pi_{\mathbb{L}} \colon \mathbb{L}(E) \to \mathbb{P}$ and $\pi_{\mathbb{P}} \colon \mathbb{P} \to N$ denote the bundle projections of the tautological line bundle and the projective bundle, respectively. We can also consider the double cover of $\mathbb{P}$ given by the sphere bundle $\mathfrak{p} \colon \mathbb{S} \to \mathbb{P}$ and the surjective submersion $\mathfrak{q} \colon E \setminus 0 \to \mathbb{S}$ to pull η back to a form $(\mathfrak{q}^!)^*(\mathfrak{p}^!)^*\eta$ on $A\big|_{E\setminus 0}$. Pick a fibre metric on E and let $\chi \in \mathscr{C}^\infty(E)$ be a function such that $\chi = 1$ in a neighbourhood of N, χ only depends on the radius and $\chi(x) = 0$ for all $x \in E$ with $|x| > 1$. Then

$$\hat{\eta} = \mathrm{d}\chi \wedge (\mathfrak{q}^!)^*(\mathfrak{p}^!)^*\eta$$

defines a closed form in $\Omega^\bullet_{\mathrm{cv}}(A\big|_E)$ that is sent to $-2(p^!)_*[\omega]$ by fibre integration, hence

$$-2(p^!)_*[\omega] = [\hat{\eta}].$$

Next, we have

$$\begin{aligned} (p^!)^*\hat{\eta} &= (p^!)^*(\mathrm{d}\chi \wedge (\mathfrak{q}^!)^*(\mathfrak{p}^!)^*\eta) \\ &= \mathrm{d}(p^*\chi) \wedge ((\mathfrak{p} \circ \mathfrak{q} \circ p)^!)^*\eta \\ &= \mathrm{d}(p^*\chi(\pi^!_{\mathbb{L}})^*\eta). \end{aligned}$$

So choosing the cut-off function in the definition of Ψ to be $p^*\chi$ and the splitting σ such that $\sigma(\tau(\pi^!_{\mathbb{L}})^*\eta) = (\pi^!_{\mathbb{L}})^*\eta$, it follows that the left square commutes and Ψ constitutes an isomorphism by the 5-Lemma. ☐

We can now complete the proof of Theorem 3.5.4.

Proof of Theorem 3.5.4, 3. We want to show that, for a closed transversal $\iota \colon N \hookrightarrow M$ of a Lie algebroid $A \Rightarrow M$ of even codimension, any Euler-like section of A along N gives rise to a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^\bullet(A) \xrightarrow{(p^!)^*} \mathrm{H}^\bullet(p^! A) \xrightarrow{(p^!)_*} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(A\big|_E) \xrightarrow{i} \mathrm{H}^{\bullet+1}(A) \longrightarrow \ldots$$

where $p \colon \mathrm{Blup}(M, N) \to M$ denotes the blowdown map of the base. Since $(p^!)^* \colon \Omega^\bullet(A) \to \Omega^\bullet(p^! A)$ is a chain map, we obtain a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^\bullet(A) \xrightarrow{(p^!)^*} \mathrm{H}^\bullet(p^! A) \xrightarrow{\tau} Q^\bullet \xrightarrow{\delta} \mathrm{H}^{\bullet+1}(A) \longrightarrow \ldots$$

In this long exact sequence, by Lemma 3.5.18 the map τ only depends on local data, i.e. writing $\iota_E \colon A\big|_E \to A$ we have $\tau_E \circ \mathrm{H}(\iota_E^*) = \tau$. Moreover, the connecting homomorphism δ is defined using any kind of splitting, so in particular we can choose one with support localised inside E. Then the result follows from (the proof of) Lemma 3.5.21. But then the rest of the statement follows from the identification

$$\mathrm{H}^\bullet(\mathrm{Blup}(A, \iota^! A)) = \mathrm{H}^\bullet(p^! A) \oplus \mathrm{H}^{\bullet-1}(\pi^!_{\mathbb{P}} A)$$

from Corollary 3.5.17. ☐

3.6 Invariant submanifolds

Suppose the orbit foliation of $A \Rightarrow M$ is singular with $N \subseteq M$ a closed and embedded singular leaf. Then one can hope to increase the dimensions of the orbit by blowing up $A\big|_N$, possibly turning $\mathrm{Blup}(A, A\big|_N)$ into a regular Lie algebroid with more easily computable cohomology. In this section we discuss two examples: The action Lie algebroids $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ and $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^3$ with singular leaves given by the origin, where in both cases we consider the coadjoint action. For $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ we find that blowing up the origin resolves the singularity, allowing to compute the cohomology of both the blowup and $\mathfrak{so}(3) \ltimes \mathbb{R}^3$, while for $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^3$ we find that even iterated blowups will not result in a regular Lie algebroid.

To compute the cohomology of $\mathrm{Blup}(A, A\big|_N)$, there exists no auxiliary statement analog to Theorem 3.5.1. Instead, one can work directly on formal cohomology, for which we can utilise the Serre spectral sequence induced by $A\big|_N$ developed in [MS24], see Section 2.2.3.

When blowing up the restriction of a Lie algebroid to an invariant submanifold, the blowdown map naturally induces a map between spectral sequences.

Lemma 3.6.1 *Let $A \Rightarrow M$ be a Lie algebroid and $N \subseteq M$ a closed and embedded invariant submanifold.*

1. *The blowdown map $p_A\colon \mathrm{Blup}(A, A\big|_N) \to A$ induces a map between Serre spectral sequences*

$$p_A^*\colon \{(E_{A|_N})_r^{\bullet,\bullet}\}_{r\geq 0} \to \{(\tilde{E}_{\mathrm{Blup}(A,A|_N)_{\mathbb{P}}})_r^{\bullet,\bullet}\}_{r\geq 0}. \tag{3.80}$$

2. *If the induced map*

$$p_A^*\colon \mathrm{H}^\bullet(A\big|_N, S^p\nu_N(M)^*) \to \mathrm{H}^\bullet(\mathrm{Blup}(A, A\big|_N)\big|_{\mathbb{P}}, S^p\nu_{\mathbb{P}}(\mathrm{Blup}(M,N))^*) \tag{3.81}$$

is an isomorphism for all $p \in \mathbb{N}_0$, then

$$p_A^*\colon \mathrm{H}^\bullet(\mathscr{J}_N^\infty\Omega^\bullet(A)) \xrightarrow{\simeq} \mathrm{H}^\bullet(\mathscr{J}_{\mathbb{P}}^\infty\Omega^\bullet(\mathrm{Blup}(A, A\big|_N))) \tag{3.82}$$

is an isomorphism.

Proof. It is easy to check that p_A^* respects the corresponding filtrations and thus induces a map between spectral sequences. Then the second part follows from Theorem 2.2.21 and the Mapping Lemma for spectral sequences [Wei94, Lemma 5.2.4]. □

3.6.1 The action Lie algebroid $\mathfrak{so}(3) \ltimes \mathbb{R}^3$

In this section we consider $A = \mathfrak{so}(3) \ltimes \mathbb{R}^3$, the action Lie algebroid corresponding to infinitesimal rotations in $\mathbb{R}^3$, which is linear around the origin. More abstractly, $A \Rightarrow \mathbb{R}^3$ is a trivial vector bundle with global frame $\{e_1, e_2, e_3\}$, and bracket and anchor given by

$$[e_i, e_j] = \sum_{k=1}^{3} \varepsilon_{ijk} e_k, \tag{3.83}$$

$$\sharp(e_i) = \sum_{j,k=1}^{3} \varepsilon_{ijk} x_j \partial_k. \tag{3.84}$$

Since $\mathfrak{so}(3)$ is compact, averaging [GW92] shows that

$$\mathrm{H}^\bullet(A) \simeq \mathrm{H}^\bullet(\mathfrak{so}(3)) \otimes \mathscr{C}^\infty(\mathbb{R}^3)^{\mathfrak{so}(3)} = \mathscr{A} \oplus 0 \oplus 0 \oplus \mathscr{A}, \tag{3.85}$$

identifying

$$\mathscr{C}^\infty(\mathbb{R}^3)^{\mathfrak{so}(3)} = \mathscr{A} := \{f \in \mathscr{C}^\infty(\mathbb{R}) : f(-x) = f(x) \text{ for all } x \in \mathbb{R}\} \tag{3.86}$$

with the even functions on $\mathbb{R}$, as

$$\mathscr{C}^\infty(\mathbb{R}^3)^{\mathfrak{so}(3)} = \{f \in \mathscr{C}^\infty(\mathbb{R}^3) : f \text{ only depends on the radius}\}. \tag{3.87}$$

Blowing up the Lie subalgebroid $A\big|_{\{0\}}$ results in yet another action Lie algebroid,

$$\mathrm{Blup}(A, A\big|_{\{0\}}) = \mathfrak{so}(3) \ltimes \mathrm{Blup}(\mathbb{R}^3, \{0\}), \tag{3.88}$$

see [Obs21, Corollary 5.93], and we write

$$\tilde{e}_i := \mathrm{Blup}(e_i) = p^\sharp e_i \in \Gamma(\mathrm{Blup}(A, A\big|_{\{0\}})). \tag{3.89}$$

We compute the cohomology of the blowup and can reproduce (3.85).

Proposition 3.6.2 *Let* $A = \mathfrak{so}(3) \ltimes \mathbb{R}^3$. *Then*

$$\mathrm{H}^\bullet(A) \simeq \mathrm{H}^\bullet(\mathfrak{so}(3) \ltimes \mathrm{Blup}(\mathbb{R}^3, \{0\})) = \mathscr{A} \oplus 0 \oplus 0 \oplus \mathscr{A}. \tag{3.90}$$

We first show the second equality by computing the cohomology of the blowup using the spectral sequence for abelian extensions and then show that the cohomology of the blowup is isomorphic to the cohomology of A by proving that the formal cohomologies are isomorphic. First we see that the blowup is an abelian extension.

Lemma 3.6.3 *The blowup* $\mathrm{Blup}(A, A\big|_{\{0\}})$ *is an abelian extension, i.e. it fits into an exact sequence of Lie algebroids*

$$0 \longrightarrow L = \ker \sharp_{\mathrm{Blup}} \longrightarrow \mathrm{Blup}(A, A\big|_{\{0\}}) \overset{\sharp_{\mathrm{Blup}}}{\longrightarrow} T\mathcal{F} \longrightarrow 0 \tag{3.91}$$

where $\mathcal{F}$ *denotes the orbit foliation of* $\mathrm{Blup}(A, A\big|_{\{0\}})$ *given by the leaves*

$$\mathcal{F} = \{p^{-1}(\{x_1^2 + x_2^2 + x_3^3 = r^2\})\}_{r \geq 0}. \tag{3.92}$$

In particular, $\mathrm{Blup}(A, A\big|_{\{0\}})$ *is a regular Lie algebroid.*

Proof. Recall that the blowdown map $p_A\colon \operatorname{Blup}(A, A\big|_{\{0\}}) \to A$ restricted to $\operatorname{Blup}(\mathbb{R}^3, \{0\}) \setminus \mathbb{P}$ is a diffeomorphism, thus, on this set the orbit foliation is the same as on $\mathbb{R}^3 \setminus \{0\}$ under p. To see that $\mathbb{P}$ is a single orbit, since the situation is highly symmetric it is enough to consider one of the charts of $\operatorname{Blup}(\mathbb{R}^3, \{0\})$ from Remark 3.3.3, say U_1. Using $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ to denote the coordinates on U_1 we find

$$\begin{aligned} \sharp_{\mathrm{Blup}}(\tilde{e}_1)\big|_{U_1} &= \tilde{x}_3\tilde{\partial}_2 - \tilde{x}_2\tilde{\partial}_3 \\ \sharp_{\mathrm{Blup}}(\tilde{e}_2)\big|_{U_1} &= -\tilde{x}_1\tilde{x}_3\tilde{\partial}_1 + \tilde{x}_2\tilde{x}_3\tilde{\partial}_2 + (1+\tilde{x}_3^2)\tilde{\partial}_3 \\ \sharp_{\mathrm{Blup}}(\tilde{e}_3)\big|_{U_1} &= \tilde{x}_1\tilde{x}_2\tilde{\partial}_1 - (1+\tilde{x}_2^2)\tilde{\partial}_2 - \tilde{x}_2\tilde{x}_3\tilde{\partial}_3. \end{aligned} \tag{3.93}$$

Thus, we see that on $U_1 \cap \mathbb{P} = \{\tilde{x}_1 = 0\}$, the image of $\sharp_{\mathrm{Blup}}$ still spans a two-dimensional distribution. Since $\mathbb{P}$ is connected, this implies that $\mathbb{P}$ is a single orbit. ☐

Thus, we are in the framework of Section 2.4. Note that in this particular case the spectral sequence will stabilise after the second page as $\operatorname{rank} L = 1$. As a vector bundle, L is given by the pullback of the tautological line bundle back to itself.

Lemma 3.6.4 *Denoting the projection of the tautological line bundle $\mathbb{L}$ of $\mathbb{P}$ by $\pi_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{P}$, the kernel of $\sharp_{\mathrm{Blup}}$ is isomorphic to*

$$L = \ker \sharp_{\mathrm{Blup}} = \pi_{\mathbb{L}}^{\sharp}\mathbb{L}, \tag{3.94}$$

by mapping

$$\pi_{\mathbb{L}}^{\sharp}\mathbb{L} \ni ((v^1, v^2, v^3)_{[v']})_{w_{[v']}} \mapsto (v^1\tilde{e}_1 + v^2\tilde{e}_2 + v^3\tilde{e}_3)\big|_{w_{[v']}} \in L. \tag{3.95}$$

Proof. The defined map (3.95) identifies $\pi_{\mathbb{L}}^{\sharp}\mathbb{L}$ with a rank 1 subbundle of $\operatorname{Blup}(A, A\big|_{\{0\}})$. Thus, we only need to show that its image lies in the kernel of $\sharp_{\mathrm{Blup}}$. In the view of the chart for U_1 of the blowup, recall that the canonical chart for U_1 already is a vector bundle chart for $\mathbb{L}_{U_1 \cap \mathbb{P}}$ with $\tilde{x}_1$ as the fibre coordinate. Here, the map (3.95) becomes

$$((\tilde{x}_1)_{[1:\tilde{x}_2:\tilde{x}_3]})_{(\tilde{x}'_1, \tilde{x}_2, \tilde{x}_3)} \mapsto \tilde{x}_1(\tilde{e}_1 + \tilde{x}_2\tilde{e}_2 + \tilde{x}_3\tilde{e}_3).$$

But given (3.93) it is easy to check that points of the form $\tilde{e}_1 + \tilde{x}_2\tilde{e}_2 + \tilde{x}_3\tilde{e}_3$ lie in the kernel of $\sharp_{\mathrm{Blup}}$. ☐

In particular, L is not a trivial vector bundle. Pulled back to a double cover, however, we can even trivialise the action of the orbit foliation.

Lemma 3.6.5 *Consider the double cover* $\operatorname{pr}\colon \mathbb{S}^2 \times \mathbb{R} \to \operatorname{Blup}(\mathbb{R}^3, \{0\})$ *defined by*

$$\operatorname{pr}\colon (x, t) \mapsto \begin{cases} p^{-1}(tx) & \text{if } t \neq 0, \\ [x] & \text{if } t = 0, \end{cases} \tag{3.96}$$

where we view $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$. Then the induced abelian extension

$$0 \longrightarrow \operatorname{pr}^! L \longrightarrow \operatorname{pr}^! \operatorname{Blup}(A, A\big|_{\{0\}}) \xrightarrow{\sharp_{\operatorname{pr}^!}} \operatorname{pr}^! T\mathcal{F} \longrightarrow 0, \tag{3.97}$$

where $\sharp_{\mathrm{pr}^!}$ *denotes the anchor of* $\mathrm{pr}^!\mathrm{Blup}(A, A|_{\{0\}})$*, has the following properties.*

1. *The Lie algebroid* $\mathrm{pr}^!\mathrm{Blup}(A, A|_{\{0\}})$ *is a product Lie algebroid*

$$\mathrm{pr}^!\mathrm{Blup}(A, A|_{\{0\}}) \simeq \mathrm{pr}^!\mathrm{Blup}(A, A|_{\{0\}})|_{\mathbb{S}^2\times\{1\}} \times (0 \Rightarrow \mathbb{R}). \tag{3.98}$$

2. *The Lie algebroid* $\mathrm{pr}^! T\mathcal{F}$ *is the foliation Lie algebroid of the foliation*

$$\{\mathbb{S}^2 \times \{t\}\}_{t\in\mathbb{R}}. \tag{3.99}$$

3. *There exists a non-vanishing section* $g \in \Gamma(\mathrm{pr}^! L)$*, anti-invariant under the action of* $\mathbb{Z}_2$*, which is constant under the action of* $\mathrm{pr}^!\mathcal{F}$*.*

Proof. For the first part, recall that

$$\mathrm{Blup}(A, A|_{\{0\}}) = \mathfrak{so}(3) \ltimes \mathrm{Blup}(\mathbb{R}^3, \{0\})$$

is an action Lie algebroid. Since $\mathrm{pr}\colon \mathbb{S}^2 \times \mathbb{R} \to \mathbb{L}$ is a double cover, we obtain

$$\mathrm{pr}^!\mathrm{Blup}(A, A|_{\{0\}}) = \mathfrak{so}(3) \ltimes (\mathbb{S}^2 \times \mathbb{R}).$$

We show that the action of $\mathfrak{so}(3)$ is independent of the $\mathbb{R}$ variable, from which the statement follows.

Note that out of charts for the blowup we get charts on $U_i^\pm = \mathrm{pr}^{-1}(U_i)$ in the natural way. Consider $(U_1^\pm, \tilde{x} \circ \mathrm{pr})$. The coordinate transformation

$$\begin{aligned} y_1 &= \tilde{x}_1\sqrt{1 + \tilde{x}_2^2 + \tilde{x}_3^2},\\ y_2 &= \tilde{x}_2,\\ y_3 &= \tilde{x}_3 \end{aligned}$$

gives a product chart for $U_1^\pm \times \mathbb{R}$, where y_1 is the coordinate on $\mathbb{R}$. In this chart, the anchor is given by

$$\begin{aligned} \sharp(\mathrm{pr}^!\tilde{e}_1) &= y_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_3},\\ \sharp(\mathrm{pr}^!\tilde{e}_2) &= -y_2y_3\frac{\partial}{\partial y_2} - (1 + y_3^2)\frac{\partial}{\partial y_3},\\ \sharp(\mathrm{pr}^!\tilde{e}_3) &= (1 + y_2^2)\frac{\partial}{\partial y_2} + y_2y_3\frac{\partial}{\partial y_3}. \end{aligned}$$

Thus, the action of $\mathfrak{so}(3)$ does not depend on the $\mathbb{R}$ variable y_1.

Using the first part, the second part is clear. For the last part, define a section of $\mathrm{pr}^!L|_{U_i^\pm}$ by $(g_i^\pm, 0)$, where

$$g_i^\pm = \frac{\pm 1}{\sqrt{1 + \sum_{j\neq i} y_j^2}}(\mathrm{pr}^!\tilde{e}_i + \sum_{j\neq i} y_j \mathrm{pr}^!\tilde{e}_j). \tag{3.100}$$

These definitions agree on overlaps and, thus, define a global trivialising section, which is constant under the action of $\mathrm{pr}^! T\mathcal{F}$ as a straightforward computation shows. ☐

This is sufficient to compute the spaces on the second page of the spectral sequence associated to the abelian extension (3.97) (Proposition 2.4.1). For the differential on the second page we show that the extension class does not vanish in such a way that $\mathrm{d}_2\colon E_2^{0,2} \to E_2^{1,0}$ is an isomorphism.

Proposition 3.6.6 *The cohomology of* $\operatorname{Blup}(A, A\big|_{\{0\}})$ *is given by*

$$\mathrm{H}^\bullet(\operatorname{Blup}(A, A\big|_{\{0\}})) \simeq \mathscr{A} \oplus 0 \oplus 0 \oplus \mathscr{A}, \tag{3.101}$$

where $\mathscr{A} = \{f \in \mathscr{C}^\infty(\mathbb{R}) \mid f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$.

Proof. By Proposition 2.4.1 and Lemma 3.6.5 the nontrivial entries on the second page of the spectral sequence associated to (3.97) are given by

$$\begin{tikzcd}
\mathscr{C}^\infty(\mathbb{R}) \arrow[drr, dashed, "\mathrm{d}_2"] & 0 & \mathscr{C}^\infty(\mathbb{R}) \\
\mathscr{C}^\infty(\mathbb{R}) & 0 & \mathscr{C}^\infty(\mathbb{R})
\end{tikzcd}$$

as $\mathrm{H}^\bullet(\mathrm{pr}^! T\mathcal{F}) = \mathscr{C}^\infty(\mathbb{R}) \oplus 0 \oplus \mathscr{C}^\infty(\mathbb{R})$ by [MS24, Lemma 5.4], see also [CM13, Lemma 3], or Theorem 2.3.5. Note that in the third column we integrated along the spheres of the foliation to identify the cohomology with functions on $\mathbb{R}$, and that the $\mathbb{Z}_2$-action reverses the orientation.

For the differential d_2 we compute the extension class (2.49). To do so, note that by Lemma 3.6.5 it is enough to consider the short exact sequence (3.97) restricted to a single fibre $\mathbb{S}^2 \times \{1\}$. We show that the extension class is given by the volume form on $\mathbb{S}^2$. Consider $\mathbb{S}^2 \subseteq \mathbb{R}^3$, identify the trivial vector bundles $\mathrm{pr}^! \operatorname{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{S}^2 \times \{1\}} \simeq T\mathbb{R}^3\big|_{\mathbb{S}^2}$, and identify for $x \in \mathbb{S}^2$ the tangent space

$$T_x\mathbb{S}^2 \simeq \{v \in \mathbb{T}_x\mathbb{R}^3 : \langle x, v\rangle = 0\} \subseteq T_x\mathbb{R}^3.$$

Note that this gives a splitting σ of the short exact sequence. To determine the curvature, note that under these identifications, the section $g \in \Gamma(\mathrm{pr}^! L)$ defined in (3.100) is given by the outward pointing unit normal vector field of $\mathbb{S}^2$. The Lie bracket on the constant sections of $T\mathbb{R}^3\big|_{\mathbb{S}^2}$ is the cross product.

For $V \in \Gamma(T\mathbb{S}^2)$, we write $\sigma(V) = \sum_{i=1}^3 v_i e_i$ for some $v_i \in \mathscr{C}^\infty(\mathbb{S}^2)$. Let $V, W \in \Gamma(T\mathbb{S}^2)$ be given. Pairing the curvature with the trivialising normal vector field yields

$$\begin{aligned}
\langle x, \gamma(V, W)\rangle &= \langle x, [\sigma(V), \sigma(W)]\rangle \\
&= \langle x, \sharp(\sigma(V)) w^\ell e_\ell - \sharp(\sigma(W)) v^\ell e_\ell\rangle + \langle x, \sigma(V) \times \sigma(W)\rangle \\
&= 3\langle x, \sigma(V) \times \sigma(W)\rangle \\
&= 3\mathrm{i}_{\sigma(W)}\mathrm{i}_{\sigma(V)}\mathrm{i}_x \mathrm{vol}_{\mathbb{R}^3},
\end{aligned}$$

since

$$
\begin{aligned}
&\sum_{\ell=1}^{3}\langle x, \sharp(\sigma(V))w_\ell e_\ell - \sharp(\sigma(W))v_\ell e_\ell\rangle \\
&\quad= \sum_{i,j,k,\ell=1}^{3} \big(\varepsilon_{ijk}x_j x_\ell(v_i(\partial_k w_\ell) - w_i(\partial_k v_\ell))\big) \\
&\quad= \sum_{i,j,k,\ell=1}^{3} \big(\varepsilon_{ijk}x_j(v_i(\partial_k \underbrace{x_\ell w_\ell}_{=0}) - v_i\delta_{k\ell}w_\ell - w_i(\partial_k \underbrace{x_\ell v_\ell}_{=0}) + w_i\delta_{kl}v_\ell)\big) \\
&\quad= \sum_{i,j,k=1}^{3} \big(\varepsilon_{ijk}x_j(-v_i w_k + w_i v_k)\big) \\
&\quad= \sum_{i,j,k=1}^{3} 2\varepsilon_{ijk}x_j w_i v_k \\
&\quad= 2\langle x, \sigma(V)\times\sigma(W)\rangle.
\end{aligned}
$$

Thus, γ is a multiple of the volume form on $\mathbb{S}^2$ and, in conclusion, wedging with the extension class of (3.97) is an isomorphism. Thus, the cohomology of $\mathrm{pr}^!\mathrm{Blup}(A, A\big|_{\{0\}})$ is given by

$$
\mathrm{H}^\bullet(\mathrm{pr}^!\mathrm{Blup}(A, A\big|_{\{0\}})) = \mathscr{C}^\infty(\mathbb{R}) \oplus 0 \oplus 0 \oplus \mathscr{C}^\infty(\mathbb{R}).
$$

Using Lemma 3.2.1 the statement follows: In degree 0 the invariant functions give the cohomology of the blowup, whereas in degree 3, since we trivialised the representation, we need anti-invariant classes of forms. But after integrating anti-invariant forms correspond to invariant functions as the $\mathbb{Z}_2$-action reverses the orientation. ☐

Next, we compute the cohomology associated to the normal representation of $\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}$ on $\nu_{\mathbb{P}}(\mathrm{Blup}(\mathbb{R}^3, \{0\})) = \mathbb{L}$. For the original Lie algebroid we know by [GW92, Theorem 3.5] that

$$
\mathrm{H}^k(A\big|_{\{0\}}, S^\ell(\mathbb{R}^3)^*) = \begin{cases} \mathbb{R} & \text{if } k = 0, 3 \text{ and } \ell \text{ even} \\ 0 & \text{otherwise.} \end{cases} \tag{3.102}
$$

We show that (3.102) is isomorphic to $\mathrm{H}^\bullet(\mathrm{Blup}(A, A\big|_{\{0\}})_{\mathbb{P}}, S^\ell\mathbb{L}^*)$, which by Lemma 3.6.1 implies that

$$
\mathrm{H}^\bullet(\mathscr{J}^\infty_{\mathbb{P}}\Omega^\bullet(\mathrm{Blup}(A, A\big|_{\{0\}}))) \simeq \mathrm{H}^\bullet(\mathscr{J}^\infty_{\{0\}}\Omega^\bullet(A)).
$$

Then the cohomology of the quotient complex

$$
\frac{\mathscr{J}^\infty_{\mathbb{P}}\Omega^\bullet(\mathrm{Blup}(A, A\big|_{\{0\}}))}{p_A^*\mathscr{J}^\infty_{\{0\}}\Omega^\bullet(A)} \tag{3.103}
$$

vanishes.

Proposition 3.6.7 *The cohomology of* $\Omega^\bullet(\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, S^\ell\mathbb{L}^*)$ *is given by*

$$\mathrm{H}^k(\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, S^\ell\mathbb{L}^*) = \begin{cases} \mathbb{R} & \textit{if } k = 0, 3 \textit{ and } \ell \textit{ even} \\ 0 & \textit{otherwise.} \end{cases} \tag{3.104}$$

In particular,

$$p_A^*\colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(\mathrm{Blup}(A, A\big|_{\{0\}})) \tag{3.105}$$

is an isomorphism.

Proof. We again make use of the double cover of $\mathbb{P}$ given by

$$\mathrm{pr}\colon \mathbb{S}^2 \ni x \mapsto [x] \in \mathbb{P}.$$

First, note that for $\ell = 1$ the constant section of the trivial bundle $\mathrm{pr}^\sharp\mathbb{L} = \mathbb{S}^2 \times \mathbb{R}$ trivialises the representation. This section is anti-invariant under the $\mathbb{Z}_2$-action. With this in mind we can now proceed as in the proof of Proposition 3.6.6 using a spectral sequence argument to compute the cohomology of $\Omega^\bullet(\mathrm{pr}^!\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, \mathrm{pr}^\sharp S^\ell\mathbb{L}^*)$. By Proposition 2.4.1 the second page of the spectral sequence is given by

$$\begin{tikzcd}
\mathbb{R} \arrow[rrd, dashed, "\mathrm{d}_2"] & 0 & \mathbb{R} \\
\mathbb{R} & 0 & \mathbb{R}
\end{tikzcd}$$

as $\mathrm{H}^\bullet(\mathbb{S}^2) = \mathbb{R} \oplus 0 \oplus \mathbb{R}$, using integration in degree 2, and the differential is again an isomorphism following a similar reasoning as in Proposition 3.6.6. Thus,

$$\mathrm{H}^\bullet(\mathrm{pr}^!\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, \mathrm{pr}^\sharp S^\ell\mathbb{L}^*) = \mathbb{R} \oplus 0 \oplus 0 \oplus \mathbb{R}.$$

Since all cohomology groups are one-dimensional, $\mathrm{H}^k(\mathrm{Blup}(A, A\big|_{\{0\}})_{\mathbb{P}}, S^\ell\mathbb{L}^*)$ will either be trivial or $\mathbb{R}$. We argue that it is nontrivial if and only if ℓ is even and $k = 0, 3$. Indeed, only in this case the trivialising section of $\mathrm{pr}^\sharp S^\ell\mathbb{L}^*$ is invariant, thus, the cohomological degree 0 part is invariant. In degree 3 the coefficients are tensored with one more copy of $\mathbb{L}^*$, but since we used integration, which is anti-invariant under the $\mathbb{Z}_2$-action, again $\mathbb{R} = \mathrm{H}^3(\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, S^\ell\mathbb{L}^*)$. If ℓ is odd, similar reasoning shows that

$$\mathrm{H}^k(\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, S^\ell\mathbb{L}^*) = 0.$$

Hence, we have

$$\mathrm{H}^k(\mathrm{Blup}(A, A\big|_{\{0\}})\big|_{\mathbb{P}}, S^\ell\mathbb{L}^*) \simeq \mathrm{H}^k(A\big|_{\{0\}}, S^\ell(\mathbb{R}^3)^*).$$

The rest follows from Lemma 3.6.1. ☐

3.6.2 A non-example: $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^3$

Consider the action Lie algebroid $A = \mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^3$ coming from the coadjoint action of $\mathfrak{sl}_2(\mathbb{R})$, which is a trivial vector bundle with global frame $\{e_1, e_2, e_3\}$ and anchor

$$\begin{aligned} \sharp(e_1) &= -x_3\partial_2 - x_2\partial_3, \\ \sharp(e_2) &= x_3\partial_1 + x_1\partial_3, \\ \sharp(e_3) &= x_2\partial_1 - x_1\partial_2. \end{aligned} \tag{3.106}$$

The bracket can be expressed in terms of the global frame, but will not be used in this section.

The orbits of this Lie algebroid are given by the connected components of the level sets of the function $f = x_1^2 + x_2^2 - x_3^2$, where $f^{-1}(\{0\})$ splits into three orbits $x_3 > 0$, $x_3 < 0$, and the origin. Hence, the origin is the only leaf of this foliation that is not two dimensional. In this section we show that by means of repeatedly blowing up restrictions of A (or its blowups) to orbits, one cannot construct a regular Lie algebroid.

Proposition 3.6.8 *The restriction of the Lie algebroid* $\tilde{A} = \mathrm{Blup}(A, A|_{\{0\}})$ *to* $\mathbb{P}$ *has of three orbits:*

- *A one-dimensional orbit*

$$Z = \{[x_1 : x_2 : x_3] \in \mathbb{P} : x_1^2 + x_2^2 = x_3^2\}; \tag{3.107}$$

- *Two two-dimensional orbits given by the connected components of* $\mathbb{P} \setminus Z$*, explicitly given by*

$$\{[x_1 : x_2 : x_3] \in \mathbb{P} : x_1^2 + x_2^2 < x_3^2\} \text{ and } \{[x_1 : x_2 : x_3] \in \mathbb{P} : x_1^2 + x_2^2 > x_3^2\}. \tag{3.108}$$

Proof. In the charts of $\mathrm{Blup}(\mathbb{R}^3, \{0\})$ from Remark 3.3.3 we can e.g. compute over $(U_1, \tilde{x})$

$$\begin{aligned} \sharp_{\mathrm{Blup}}(p^\sharp e_1)\big|_{U_1} &= -(\tilde{x}_3\tilde{\partial}_2 + \tilde{x}_2\tilde{\partial}_3), \\ \sharp_{\mathrm{Blup}}(p^\sharp e_2)\big|_{U_1} &= \tilde{x}_1\tilde{x}_3\tilde{\partial}_1 - \tilde{x}_2\tilde{x}_3\tilde{\partial}_2 + (1 - \tilde{x}_3^2)\tilde{\partial}_3, \\ \sharp_{\mathrm{Blup}}(p^\sharp e_3)\big|_{U_1} &= \tilde{x}_1\tilde{x}_2\tilde{\partial}_1 - (1 + \tilde{x}_2^2)\tilde{\partial}_2 - \tilde{x}_2\tilde{x}_3\tilde{\partial}_3. \end{aligned}$$

Thus, on the invariant submanifold $\mathbb{P} \cap U_1$, the orbit foliation is spanned by

$$\{\underbrace{\tilde{x}_3\tilde{\partial}_2 + \tilde{x}_2\tilde{\partial}_3}_{(I)}, \underbrace{(1 + \tilde{x}_2^2)\tilde{\partial}_2 + \tilde{x}_2\tilde{x}_3\tilde{\partial}_3}_{(II)}, \underbrace{\tilde{x}_2\tilde{x}_3\tilde{\partial}_2 - (1 - \tilde{x}_3^2)\tilde{\partial}_3}_{(III)}\}.$$

If $\tilde{x}_3 = 0$, this equals the span of $\{\tilde{\partial}_2, \tilde{\partial}_3\}$, which is two-dimensional. If $\tilde{x}_3 \neq 0$, (I) and (II) span a two-dimensional subspace as long as $1 + \tilde{x}_2^2 - \tilde{x}_3^2 \neq 0$ as

$$\det\begin{pmatrix} \tilde{x}_2 & \tilde{x}_3 \\ (1 - \tilde{x}_3^2) & -\tilde{x}_2\tilde{x}_3 \end{pmatrix} = -\tilde{x}_3(1 + \tilde{x}_2^2 - \tilde{x}_3^2).$$

If $\tilde{x}_2 = 0$ and $\tilde{x}_3 = 1$, i.e. it is a point in Z, the span clearly is one-dimensional. Thus, let $\tilde{x}_2 \neq 0$ and $1 + \tilde{x}_2^2 = \tilde{x}_3^2$. Then

$$\begin{aligned} \tilde{x}_3(II) + \tilde{x}_2(I) &= \tilde{x}_3\tilde{\partial}_2 + \tilde{x}_2\tilde{\partial}_3, \\ \tilde{x}_3(II) - \tilde{x}_2(I) &= \tilde{x}_3(\tilde{x}_2^2 + \tilde{x}_3^2)\tilde{\partial}_2 + \tilde{x}_2(\tilde{x}_2^2 + \tilde{x}_3^2)\tilde{\partial}_3, \end{aligned} \tag{3.109}$$

both of which are multiples of $\tilde{x}_3\tilde{\partial}_2 + \tilde{x}_2\tilde{\partial}_3$, which shows that the image of $\sharp$ over points in $Z \cap U_1$ is one-dimensional. Similar computations for the other charts show that the orbit foliation is indeed as stated. □

Thus, by blowing up the foliation has become less singular. Since Z is completely contained in $U = U_3 \setminus \{[0:0:1]\}$, in showing that the remaining singularity cannot be resolved in the sense that always at least one orbit will be one-dimensional, we can restrict the discussion to this open subset. Firstly, in this chart let us introduce polar coordinates for $(\tilde{x}_1, \tilde{x}_2)$ by

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = ((r+1)\cos\phi, (r+1)\sin\phi, \tilde{x}_3), \tag{3.110}$$

where $r \in (-1, \infty)$.

Lemma 3.6.9 *Over U the image of the anchor map is generated by*

$$\{\partial_\phi, r(r+2)\partial_r - (r-1)\tilde{x}_3\tilde{\partial}_3\}. \tag{3.111}$$

Proof. Over U_3, the anchor of the blowup maps

$$\begin{aligned} \sharp_{\mathrm{Blup}}(p^\sharp e_1)\big|_{U_3} &= \tilde{x}_1\tilde{x}_2\tilde{\partial}_1 + (\tilde{x}_2^2 - 1)\tilde{\partial}_2 - \tilde{x}_2\tilde{x}_3\tilde{\partial}_3, \\ \sharp_{\mathrm{Blup}}(p^\sharp e_2)\big|_{U_3} &= (1 - \tilde{x}_1^2)\tilde{\partial}_1 - \tilde{x}_1\tilde{x}_2\tilde{\partial}_2 + \tilde{x}_1\tilde{x}_3\tilde{\partial}_3, \\ \sharp_{\mathrm{Blup}}(p^\sharp e_3)\big|_{U_3} &= \tilde{x}_2\tilde{\partial}_1 - \tilde{x}_1\tilde{\partial}_2. \end{aligned}$$

The change of coordinates leads to

$$\begin{aligned} \sharp_{\mathrm{Blup}}(p^\sharp e_1)\big|_U &= r(r+2)\sin(\phi)\partial_r - \tfrac{1}{r+1}\cos(\phi)\partial_\phi - (r+1)\tilde{x}_3\sin(\phi)\tilde{\partial}_3, \\ \sharp_{\mathrm{Blup}}(p^\sharp e_2)\big|_U &= -r(r+2)\cos(\phi)\partial_r - \tfrac{1}{r+1}\sin(\phi)\partial_\phi + (r+1)\tilde{x}_3\cos(\phi)\tilde{\partial}_3, \\ \sharp_{\mathrm{Blup}}(p^\sharp e_3)\big|_U &= -\partial_\phi, \end{aligned}$$

which implies the statement. Indeed, the collection $\{f_1, f_2, f_3\}$ with $f_1 = \sin(\phi)p^\sharp e_1 - \cos(\phi)p^\sharp e_2$, $f_2 = \cos(\phi)p^\sharp e_1 + \sin(\phi)p^\sharp e_2$, and $f_3 = p^\sharp e_3$ constitutes a frame over U, where f_3 and f_1 are mapped to (3.111), while $\sharp_{\mathrm{Blup}}(f_2)$ is a multiple of ∂_ϕ. □

Setting $s = \frac{r}{(r+2)^3}$ one can simplify (3.111) in the sense that the image of the anchor is then spanned by

$$\{\partial_\phi, 2s\partial_s + \tilde{x}_3\tilde{\partial}_3\}. \tag{3.112}$$

Remark 3.6.10 From Lemma 3.6.9 we also find that $\tilde{A}_{\mathbb{P}}$ is an abelian extension

$$0 \longrightarrow L \longrightarrow \tilde{A}\big|_{\mathbb{P}} \longrightarrow \mathrm{Blup}(T\mathbb{P}, TZ) \longrightarrow 0 \tag{3.113}$$

where $(L \subseteq \ker \sharp_{\mathrm{Blup}})\big|_{\mathbb{P}}$ is an abelian Lie algebroid of rank 1.

However, the generating sections (3.112) of $\sharp_{\mathrm{Blup}}(\tilde{A}|_U)$ show that we cannot remove the singularity by blowing up further. There will always remain at least one leaf of dimension 1.

Lemma 3.6.11 *Let $U \subseteq \mathbb{R}^n$ be open with $0 \in U$ and $a_1, \ldots, a_n \in \mathscr{C}^\infty(U)$. Then the section*

$$\sum_i x_i a_i \partial_i \in \Gamma(TU) \tag{3.114}$$

induces a section $s \in \Gamma(\mathrm{Blup}(TU, \{0\}))$ of the Lie algebroid blowup, and $\sharp_{\mathrm{Blup}}(s)$ vanishes on the subset $Z = \{[1:0:\cdots:0], \ldots, [0:\cdots:0:1]\}$.

Moreover, around points in Z the vector field $\sharp_{\mathrm{Blup}}(s)$ is again of the form (3.114)*.*

Proof. Fix $i \in \{1, \ldots, n\}$. Writing $p\colon \mathrm{Blup}(U, \{0\}) \to U$ for the blowdown map, by Lemma 3.3.4 we have inside the chart $(U_i, \tilde{x})$

$$\sharp_{\mathrm{Blup}}(s)|_{U_i} = \tilde{x}_i p^*(a_i)\tilde{\partial}_i + \sum_{j \neq i} \tilde{x}_j (p^*(a_j) - p^*(a_i))\tilde{\partial}_j.$$

This section vanishes in the origin of U_i, i.e. the point $[0 :, \ldots, : \underbrace{1}_{i} : \cdots : 0]$, and is of the form (3.114). □

Corollary 3.6.12 *Repeatedly blowing up the one-dimensional orbits will not result in a Lie algebroid that has a* 2*-dimensional orbit foliation. Every blowup along a single* 1*-dimensional orbit will increase the number of* 1*-dimensional orbits by* 1*.*

Proof. After the first blowup the statement follows inductively from Lemma 3.6.11. Indeed, considering the sections (3.112) that generate the image of the anchor map around the one-dimensional orbit, the section $2s\partial_s + \tilde{x}_3\tilde{\partial}_3$ satisfies the requirements of Lemma 3.6.11. □

In conclusion, blowing up further will only increase the number of singularities.

3.7 Appendix A - Characterisation of $p^*\Gamma(E^*) \subseteq \Gamma(\mathrm{Blup}(E, E|_N))$

We aim to fully characterise the subset $p^*\Gamma(E^*) \subseteq \Gamma(\mathrm{Blup}(E, E|_N))$, where $E \to M$ is a vector bundle and $N \subseteq M$ a closed and embedded submanifold. We first discuss the case $E = M \times \mathbb{R} \to M$, i.e. $\Gamma(E^*) \simeq \mathscr{C}^\infty(M)$. A function $\tilde{f} \in \mathscr{C}^\infty(\mathrm{Blup}(M, N))$ in the image of $p^*\colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(\mathrm{Blup}(M, N))$ is necessarily constant on the projective bundle $\mathbb{P} = p^{-1}(N)$. However, even in the simplest non-trivial case, not every smooth function on the $\mathrm{Blup}(M, N)$ that is constant on $\mathbb{P}$ is given by a pullback.

Example 3.7.1 Consider $\mathrm{Blup}(\mathbb{R}^2, \{0\})$. Let $\tilde{g}\colon \mathbb{RP}^1 \to \mathbb{R}$ be any smooth, non-constant function. Then

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x^n \tilde{g}([x : y]) \end{aligned} \tag{3.115}$$

is continuous on $\mathbb{R}^2$, smooth on $\mathbb{R}^2 \setminus \{0\}$, homogeneous of degree n, but not a polynomial. Hence, it cannot be smooth. More precisely, one can show that $f \in \mathscr{C}^{n-1}(\mathbb{R}^2)$ is $n-1$-times continuously differentiable. However, $p^*f = p^*(x^n)p^*\tilde{g} \in \mathscr{C}^\infty(\mathrm{Blup}(M, N))$, and $p^*f\big|_{\mathbb{P}} = 0$ is constant.

However, given $\tilde{f} \in \mathscr{C}^\infty(\mathrm{Blup}(M, N))$ with $\tilde{f}\big|_{\mathbb{P}} = \text{const.}$, one can set-theoretically define a function f on M that satisfies $p^*\tilde{f} = f$ and is always *continuous*. Indeed, continuity follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Blup}(M, N) \arrow[d, "p"'] \arrow[dr, "f"] & \\ M \arrow[r, "\tilde{f}"] & \mathbb{R} \end{tikzcd} \tag{3.116}$$

and the universal property of the quotient topology. The following theorem determines the differentiability of the function $f\colon M \to \mathbb{R}$.

Lemma 3.7.2 *Let $N \subseteq M$ be a closed and embedded submanifold and $\tilde{f} \in \mathscr{C}^\infty(\mathrm{Blup}(M, N))$. Let $E \to N$ be a tubular neighbourhood of N in M, $n \in \mathbb{N}_0 \cup \{\infty\}$ and denote the blowdown map by $p\colon \mathrm{Blup}(M, N) \to M$. Then the following are equivalent.*

(a) There exists $f \in \mathscr{C}^n(M)$ with $p^*f = \tilde{f}$;

(b) For all $k \leq n$ the functions

$$\begin{aligned} f_k\colon E \setminus N &\to \mathbb{R} \\ v &\mapsto \tfrac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big|_{\lambda=0} \tilde{f}(p^{-1}(\lambda v)) \end{aligned} \tag{3.117}$$

extend smoothly to E.

Proof. Note that to prove the lemma, we only need to determine the order of differentiability of the continuous function defined by (3.116) in points on N. But in the direction of N the function clearly is smooth, so without loss of generality we can restrict to the case of $N = \{0\} \subseteq \mathbb{R}^N = M$. Assume (a). Then, for $v \in \mathbb{R}^N \setminus \{0\}$,

$$f_k(v) = \tfrac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big|_{\lambda=0} \tilde{f}(p^{-1}(\lambda v)) = \tfrac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big|_{\lambda=0} f(\lambda v),$$

hence $f_k \in \mathrm{Pol}^k(\mathbb{R}^N)$ is the highest degree part of the k-th Taylor polynomial of f and clearly extends smoothly into 0, i.e. (b) holds. For the converse, note that by Borel's Lemma we can find a function $g \in \mathscr{C}^\infty(\mathbb{R}^N)$ so that its n-th Taylor polynomial is given by $\sum_k f_k$. Then $\hat{f} = \tilde{f} - p^*g$ vanishes to n-th order along $\mathbb{P}$ and $\hat{f} \in p^*\mathscr{C}^n(\mathbb{R}^N)$ iff $\tilde{f} \in p^*\mathscr{C}^n(\mathbb{R}^N)$. Hence, we can assume that $f_k = 0$ for all $k \leq n$. Using a similar reasoning as in the proof of Lemma 3.4.4 one can then inductively show the statement. Indeed, let $v \in \mathbb{R}^N \backslash \{0\}$ be given and denote by ∂_v the directional derivative in direction v, i.e. $\partial_v = \sum_j v_j \frac{\partial}{\partial x_j}$. Let $y \in \mathbb{R}^N$ with $y_i \neq 0$. Then, using the chart $(U_i, \tilde{x})$ of the blowup from Remark 3.3.3, we have

$$\partial_v f(y) = \Big(v_i \frac{\partial \tilde{f}}{\partial \tilde{x}_i} + \sum_{j \neq i} \frac{1}{\tilde{x}_i}(v_j - v_i \tilde{x}_j) \frac{\partial \tilde{f}}{\partial \tilde{x}_j}\Big)(p^{-1}(y)).$$

The functions $\Big(v_i \frac{\partial \tilde{f}}{\partial \tilde{x}_i} + \sum_{j \neq i} \frac{1}{\tilde{x}_i}(v_j - v_i \tilde{x}_j) \frac{\partial \tilde{f}}{\partial \tilde{x}_j}\Big)$, for different $(U_i, \tilde{x})$, patch together to a global function on $\operatorname{Blup}(\mathbb{R}^N, \{0\})$ that vanishes on $\mathbb{P}$ to order $n-1$. Hence, inductively, the implication $(b) \implies (a)$ follows. $\square$

Using the notion of jets we can rephrase Lemma 3.7.2. Recall that $p^*\mathcal{I}_N \subseteq \mathcal{I}_{\mathbb{P}}$, hence there is an induced map

$$p^* \colon \mathscr{J}^k_N(\mathscr{C}^\infty(M)) \to \mathscr{J}^k_{\mathbb{P}}(\mathscr{C}^\infty(\operatorname{Blup}(M, N))). \tag{3.118}$$

Writing $j^k_N \colon \mathscr{C}^\infty(M) \to \mathscr{J}^k_N(\mathscr{C}^\infty(M))$ for the projection, Lemma 3.7.2 reads the following.

Lemma 3.7.3 *Let $N \subseteq M$ be a closed and embedded submanifold, fix $n \in \mathbb{N}_0 \cup \{\infty\}$, and let $\tilde{f} \in \mathscr{C}^\infty(\operatorname{Blup}(M, N))$ be given. Then the following are equivalent.*

(a) There exists $f \in \mathscr{C}^n(M)$ with $p^* f = \tilde{f}$;

(b) There exists $g \in \mathscr{C}^\infty(M)$ such that

$$p^* j^n_N(g) = j^n_{\mathbb{P}}(\tilde{f}). \tag{3.119}$$

From Lemma 3.7.3 the corresponding statement for sections of a vector bundle follows immediately.

Theorem 3.7.4 *Let $E \to M$ be a vector bundle and $N \subseteq M$ a closed and embedded submanifold. Let $\tilde{\alpha} \in \Gamma(\operatorname{Blup}(E, E\big|_N))$ and $n \in \mathbb{N}_0 \cup \{\infty\}$. Then the following are equivalent.*

(a) There exists $\alpha \in \Gamma^n(E^*)$ with $p^*\alpha = \tilde{\alpha}$;

(b) There exists $\beta \in \Gamma(E^*)$ such that

$$p^* j^n_N(\beta) = j^n_{\mathbb{P}}(\tilde{\alpha}). \tag{3.120}$$

3.8 Appendix B - A Gysin sequence for Lie algebroids

We develop a Gysin sequence for Lie algebroids, which we made use of in Section 3.5. To be able to formulate the result, we first introduce a notion of fibre integration for Lie algebroids.

3.8.1 Integration along fibres

Throughout this section, let $\mathrm{pr}\colon F \to N$ denote a locally trivial fibre bundle with orientable and connected fibres and $B \Rightarrow N$ a Lie algebroid. We aim to define a notion of integrating along fibres

$$(\pi^!)_*\colon \mathrm{H}^\bullet_{\mathrm{cv}}(\pi^! B) \to \mathrm{H}^{\bullet - \operatorname{rank} F}(B, o(F)), \tag{3.121}$$

where $\mathrm{H}^\bullet_{\mathrm{cv}}(\pi^! B)$ denotes compact vertical cohomology, see Definition 3.5.3. Here, $o(F) \to N$ denotes the orientation bundle of F. This is standard for vector bundles, but can also be made sense of in our more general situation. Since the fibres are orientable we can construct a double cover $\tilde{N}$ of N by

$$\tilde{N}_p = \{\lambda : \lambda \text{ is an orientation on the manifold } F_p\} \tag{3.122}$$

with the obvious smooth structure, which we call the **orientation double cover**. On the trivial bundle $\tilde{N} \times \mathbb{R} \to \tilde{N}$ we have a $\mathbb{Z}_2$ action given by

$$\hat{1}.(\lambda, t) = (\bar{\lambda}, -t), \tag{3.123}$$

where $\bar{\lambda}$ denotes the opposite orientation. Then

$$o(F) = \tilde{N} \times \mathbb{R}/\mathbb{Z}_2 \to N. \tag{3.124}$$

The bundle (3.124) is trivial if and only if there is a globally consistent way of choosing orientations for the fibres of F. In this case, we call the fibre bundle $F \to N$ **orientable**.

Remark 3.8.1 If $F \to N$ is a fibre bundle such that $o(F)$ is not the trivial bundle, pulling back F to $\tilde{N}$ will result in an orientable fibre bundle.

The orientation bundle carries a canonical flat connection of TN (induced by the trivial $T\tilde{N}$-connection on $\tilde{N} \times \mathbb{R}$) analogous to [BT82, Chapter 7] in the case of vector bundles, and the representation of B on $o(F)$ is induced by this connection using the anchor map.

To define fibre integration, we first consider the case $B = 0$, i.e. $\pi^! B = \mathcal{F}(\pi)$ is the Lie algebroid corresponding to the foliation of F into the fibres of π.

Definition 3.8.2 Let $\pi\colon F \to N$ be a locally trivial fibre bundle of rank k with orientable and connected fibres. Then fibre integration on $\Omega^\bullet_{\mathrm{cv}}(\mathcal{F}(\pi))$ is defined by

$$\int_{\mathcal{F}(\pi)} \omega = \begin{cases} \left(N \ni p \mapsto \int_{F_p} \iota_p^* \omega\right) \in \Gamma(o(F)) & \text{if } \omega \in \Omega^k_{\mathrm{cv}}(\mathcal{F}(\pi)) \\ 0 & \text{otherwise.} \end{cases} \tag{3.125}$$

Here, $\iota_p\colon F_p \to F$ denotes the inclusion of the fibre at $p \in N$.

Remark 3.8.3 Fibre integration is well-defined and coincides with the standard notion of fibre integration in the following sense. Since $\mathcal{F}(\pi) \hookrightarrow TF$ we can choose a vector bundle complement (i.e. a horizontal subbundle) C, i.e. $TF = \mathcal{F}(\pi) \oplus C$, to extend foliated forms to forms on TF. The result of the ordinary fibre integration will not depend on the chosen complement (since it is a top-degree form of $\mathcal{F}(\pi)$) and coincides with (3.125).

Proposition 3.8.4 *Fibre integration descends to a map*

$$\int_{\mathcal{F}(\pi)} : \mathrm{H}^k_{\mathrm{cv}}(\mathcal{F}(\pi)) \to \Gamma(o(F)). \tag{3.126}$$

Proof. We only need to check that integration vanishes on exact forms. But, since we have for foliated forms $\omega \in \Omega^\bullet(\mathcal{F}(\pi))$ that

$$\mathrm{d}_{F_p} \iota_p^* \omega = \iota_p^* \mathrm{d}_{\mathcal{F}} \omega,$$

this follows from Stoke's theorem. □

Recall that in case of an orientable vector bundle $F \to N$, ordinary fibre integration of differential forms yields an isomorphism

$$\mathrm{H}^\bullet_{\mathrm{cv}}(F) \simeq \mathrm{H}^{\bullet - k}(N), \tag{3.127}$$

where the pre-image of $1 \in \mathrm{H}^0(N)$ is the **Thom class** $\theta \in \mathrm{H}^k_{\mathrm{cv}}(F)$ (see [BT82, (12.2.1)]). In our setting we obtain a similar statement in Lemma 3.8.9, which in the current situation is the following.

Lemma 3.8.5 *Let $\pi \colon F \to N$ be a vector bundle of rank k. Then*

$$\mathrm{H}^n_{\mathrm{cv}}(\mathcal{F}(\pi)) \simeq \begin{cases} \Gamma(o(F)) & \text{if } n = k \\ 0 & \text{otherwise,} \end{cases} \tag{3.128}$$

where we identify the $\mathscr{C}^\infty(N)$-modules

$$\int_{\mathcal{F}(\pi)} : \mathrm{H}^k_{\mathrm{cv}}(\mathcal{F}(\pi)) \xrightarrow{\simeq} \Gamma(o(F)). \tag{3.129}$$

If F is oriented, for $1 \in \mathscr{C}^\infty(N) = \Gamma(o(F))$ we have

$$\left(\int_{\mathcal{F}(\pi)} \right)^{-1} (1) = \sharp_{\mathcal{F}}^* \theta, \tag{3.130}$$

where $\theta \in \mathrm{H}^k_{\mathrm{cv}}(F)$ denotes the Thom class and $\sharp_{\mathcal{F}}$ denotes the anchor of $\mathcal{F}(\pi)$.

Proof. We first show (3.129). Note that integrating is indeed a map of $\mathscr{C}^\infty(N)$-modules, where the module structure on $\mathrm{H}^\bullet_{\mathrm{cv}}(\mathcal{F}(\pi))$ is given by multiplying with pullbacks. The statement in the non-orientable case follows from the orientable one by pulling everything back to a double cover $\tilde{N} \to N$ which trivialises $o(F)$. Thus, suppose F is orientable and let $\theta \in \mathrm{H}^k_{\mathrm{cv}}(F)$ denote the Thom class of the orientable vector bundle. We can pick any representative of the Thom class to compute its integral, and for the computation we only need the contributions that are k-tangent to the foliation since the rest is mapped to zero anyway. But those are given by $\sharp_{\mathcal{F}}^* \theta$, so we get

$$\int_{\mathcal{F}(\pi)} \sharp_{\mathcal{F}}^* \theta = 1 \in \mathscr{C}^\infty(N)$$

immediately. Since integrating is a module morphism, this implies surjectivity. For injectivity it is enough to argue locally, since we can exploit the $\mathscr{C}^\infty(N)$-module structure and the existence of a partition of unity on N. Thus, let $F = N \times \mathbb{R}^k$ with coordinates (x, y) and

$$\omega = f \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_k \in \Omega^k_{\mathrm{cv}}(\mathcal{F}(\pi))$$

be given, where $f \in \mathscr{C}^\infty_{\mathrm{cv}}(F)$. Suppose that $\int_{\mathcal{F}(\pi)} \omega = 0$, i.e.

$$\int f(x, y)\mathrm{d}y = 0$$

for every $x \in N$. One can adapt the proof of [GR02, Lemma 2.4] to show that there exist functions $g_1, \dots, g_k \in \mathscr{C}^\infty_{\mathrm{cv}}(F)$ such that

$$f = \sum_{i=1}^{k} \frac{\partial g_i}{\partial y_i}.$$

Then $\eta \in \Omega^{k-1}_{\mathrm{cv}}(\mathcal{F}(\pi))$ defined by

$$\eta = \sum_{i=1}^{k} (-1)^{i+1} g_i \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_{i-1} \wedge \mathrm{d}y_{i+1} \wedge \dots \wedge \mathrm{d}y_k$$

is a primitive for ω.

To show that $\mathrm{H}^n_{\mathrm{cv}}(\mathcal{F}(\pi)) = 0$ if $n \neq k$, first note that for $n = 0$ the statement is clear. For $1 \leq n < k$ note that again $\mathrm{H}^n_{\mathrm{cv}}(\mathcal{F}(\pi))$ is a $\mathscr{C}^\infty(N)$-module. Hence, it is enough to show that the cohomology vanishes over a small enough open subsets of N, where we utilise the proof of the well-known Poincaré-Lemma for compact support. Let $U \subseteq N$ be relatively compact such that F is trivial over an open neighbourhood of the closure of U. Let $\omega \in \Omega^n_{\mathrm{cv}}(\mathcal{F}(\pi))$ be closed. By definition of $\Omega^n_{\mathrm{cv}}(\mathcal{F}(\pi))$ and by compactness of K there exists a compact subset $A \subseteq \mathbb{R}^k$ such that

$$\operatorname{supp}(\omega)\big|_{\pi^{-1}(U)} \subseteq A \times U.$$

Denote by $i \colon \mathbb{R}^k \hookrightarrow \mathbb{S}^k$ the embedding of $\mathbb{R}^k$ into the k-sphere via the stereographic projection of the north pole $\mathcal{N}$, and by

$$i_* \colon \Omega^n_{\mathrm{cv}}(\mathcal{F}(\pi)\big|_{\pi^{-1}(U)}) \to \Omega^n_{\mathcal{F}}(\mathbb{S}^k \times U)$$

the extension by 0 as a foliated form on $S^k \times U \to U$. Then, since $\mathrm{H}^n_{\mathcal{F}}(\mathbb{S}^k \times U) = 0$ by [MS24, Lemma 4.8], and $\mathrm{d}i_*\omega = 0$, there exists $\eta \in \Omega^{n-1}_{\mathcal{F}}(\mathbb{S}^k \times U)$ with $\mathrm{d}_{\mathcal{F}}\eta = i_*\omega$. Let $O \subseteq \mathbb{S}^k \setminus i(A)$ be a contractible neighbourhood of $\mathcal{N}$ and χ a bump function which is 1 in a neighbourhood of $\mathcal{N}$ and supported on O. On $O \times U$, we have

$$\mathrm{d}_{\mathcal{F}}\eta\big|_{O \times U} = i_*\omega\big|_{O \times U} = 0. \tag{$*$}$$

Thus, if $n > 1$, there exists $\tilde{\eta} \in \Omega^{n-2}(O \times U)$ with $\mathrm{d}_{\mathcal{F}}\tilde{\eta} = \eta\big|_{O \times U}$, again by [MS24, Lemma 4.8]. Then

$$i^*(\eta - \mathrm{d}_{\mathcal{F}}(\chi\tilde{\eta})) \in \Omega^{n-1}_{\mathrm{cv}}(\mathbb{R}^k \times U)$$

is a well-defined primitive of ω.

If $n = 1$, then $(*)$ implies that $\eta\big|_{O\times U}$ is the pullback of a function $f \in \mathscr{C}^\infty(U)$. Hence, in this case $i^*(\eta - \pi^*_{\mathbb{S}^k\times U} f) \in \mathscr{C}^\infty_{\mathrm{cv}}(\mathbb{R}^k \times U)$ is a primitive of ω. ☐

If the fibres of $\pi\colon F \to N$ are compact, orientable, and connected, we obtain a similar statement for the top degree foliated cohomology.

Lemma 3.8.6 *Let $\pi\colon F \to N$ be a locally trivial fibre bundle with typical fibre a compact, orientable, and connected manifold of dimension k. Then fibre integration yields an isomorphism*

$$\int_{\mathcal{F}(\pi)} : \mathrm{H}^k(\mathcal{F}(\pi)) \simeq \Gamma(o(F)) \tag{3.131}$$

Proof. By [MS24, Lemma 4.8] (see Lemma 2.3.8), the foliated cohomology $\mathrm{H}^k(\mathcal{F}(\pi))$ is given by sections of a line bundle over N with fibres $\mathrm{H}^k(\pi^{-1}(x))$ for all $x \in N$, which can be readily identified with $o(F)$ via fibre integration. ☐

This concludes the discussion for $\pi^!B$, where $B \Rightarrow N$ is the zero Lie algebroid. For a general Lie algebroid we can first decompose the forms on $\pi^!B$ according to their vertical part.

Lemma 3.8.7 *Let $F \to N$ be a locally trivial fibre bundle and $B \Rightarrow N$ a Lie algebroid. Picking a connection on F leads to a decomposition*

$$\pi^!B = \mathrm{Ver}(F) \oplus \pi^\sharp B \tag{3.132}$$

with the property that the anchor is given by the identity on vertical vectors and maps $\pi^\sharp a \mapsto \sharp(a)^{\mathrm{hor}}$ for $a \in \Gamma(B)$, and $\pi^!\colon \pi^!B \to B$ is given by $\pi^! = \pi^\sharp \circ (\mathrm{pr}_{\pi^!B\to\pi^\sharp B})$.

Proof. Given a connection on E and local frames $\{s_\alpha\}_\alpha$ of $F\big|_U$ and $\{a_i\}_i$ of $B\big|_U$ for some open $U \subseteq N$, the collection

$$\{(0, s_\alpha^{\mathrm{ver}})\}_\alpha \cup \{(\pi^\sharp a, \sharp(a_i)^{\mathrm{hor}})\}_i$$

yields a local frame for $\pi^!B\big|_{F|_U}$. Then the statements follow immediately. ☐

Thus, by choosing a connection on F we obtain a decomposition

$$\Omega^\bullet_{\mathrm{cv}}(\pi^!B) = \bigoplus_{p+q=\bullet} \Omega^p_{\mathrm{cv}}(\mathcal{F}(\pi), \pi^\sharp\Lambda^q B^*). \tag{3.133}$$

This allows to define fibre integration for forms on $\pi^!B$ by just integrating out the fibre components. More precisely, for $\omega = \sum_{i,j} \eta_{i,j} \otimes \pi^\sharp\alpha_{i,j}$, where $\eta_{i,j} \in \Omega^j_{\mathrm{cv}}(\mathcal{F}(\pi))$ and $\alpha_{i,j} \in \Omega^\bullet(B)$, we define

$$(\pi^!)_*\Big(\sum_{i,j} \eta_{i,j} \otimes \pi^\sharp\alpha_{i,j}\Big) = \sum_i \Big(\int_{\mathcal{F}(\pi)} \eta_{i,k}\Big)\alpha_{i,k}. \tag{3.134}$$

It is clear that (3.134) yields a map

$$(\pi^!)_* \colon \Omega^\bullet_{\mathrm{cv}}(\pi^! B) \to \Omega^{\bullet - k}(B, o(F)). \tag{3.135}$$

Since only contributions of $\Omega^k(\mathcal{F}(\pi))$ (i.e. of top degree) matter in computing the integral, it does not depend on the chosen connection, which is also emphasised by the following description. Consider $\omega \in \Omega^\bullet_{\mathrm{cv}}(\pi^! B)$ and fix $p \in N$. Then we can define the **k-fold restriction** $\omega\big|^k_{F_p} \in \Omega^k_c(F_p, \Lambda^{\bullet-k} B^*_p)$ of ω to F_p in the following way: For $X_1, \dots, X_k \in \Gamma(TF_p)$ we define

$$\omega\big|^k_{F_p}(X_1, \dots, X_k) \colon F_p \to \Lambda^{\bullet-k}(\pi^\sharp B^*)\big|_{F_p} = \Lambda^{\bullet-k}(B^*_p) \tag{3.136}$$

in the obvious way, as $\pi^! B / \mathcal{F}(\pi) = \pi^\sharp B$. Then one integrates this form along F_p, which yields the same result as (3.134) (note that the integral will vanish if ω is not of foliated degree k).

Lemma 3.8.8 *Let $F \to N$ be a locally trivial fibre bundle of rank k with orientable and connected fibres with trivial orientation bundle, and $B \Rightarrow N$ a Lie algebroid. Then $(\pi^!)_* \colon \Omega^\bullet_{\mathrm{cv}}(\pi^! B) \to \Omega^{\bullet-k}(B)$ is, up to a sign depending on the degree, a morphism of cochain complexes and induces a map*

$$(\pi^!)_* \colon \mathrm{H}^\bullet_{\mathrm{cv}}(\pi^! B) \to \mathrm{H}^{\bullet-k}(B). \tag{3.137}$$

Proof. Since $(\pi^!)_*$, $\mathrm{d}_{\pi^! B}$ and d_B are local in the sense that to calculate them in a point $p \in N$ we only need information about the form on $F\big|_U$ and U, respectively, where U is a neighbourhood of p. Thus, suppose that $\pi \colon F = \tilde{F} \times N \to N$ is a product bundle. Then the canonical flat connection induces a decomposition

$$\Omega^n_{\mathrm{cv}}(\pi^! B) \simeq \bigoplus_{i+j=n} \Omega^i_{\mathrm{cv}}(\mathcal{F}(\pi)) \otimes_{\mathscr{C}^\infty(B)} \Omega^j(B).$$

Under this decomposition, we obtain the following.

1. By flatness of the connection the differential on $\Omega^n_{\mathrm{cv}}(\pi^! B)$ splits into
$$\mathrm{d}_{\pi^! B} = \mathrm{d}_{\mathcal{F}(\pi)} \otimes \mathrm{id} + (-1)^j \mathrm{id} \otimes \mathrm{d}_B.$$
2. By definition of $(\pi^!)_*$ we have for $\omega \in \Omega^\bullet_{\mathrm{cv}}(\mathcal{F}(\pi))$ and $\alpha \in \Omega^\bullet(B)$ that
$$(\pi^!)_*(\omega \otimes (\pi^!)^* \alpha) = ((\pi^!)_* \omega) \otimes (\pi^!)^* \alpha.$$
3. By Lemma 3.8.5 we have
$$(\pi^!)_* \mathrm{d}_{\mathcal{F}(\pi)} \Omega^\bullet_{\mathrm{cv}}(\mathcal{F}(\pi)) = 0.$$

Together, this implies the statement. □

With the fibre integration on cohomology defined, we get a Thom isomorphism for Lie algebroids as a consequence of Lemma 3.8.5.

Lemma 3.8.9 (Thom isomorphism for Lie algebroids) *Let $\pi\colon E \to N$ be a vector bundle of rank k with orientation bundle $o(E)$ and $B \Rightarrow N$ a Lie algebroid. Then fibre integration*

$$(\pi^!)_*\colon \mathrm{H}^\bullet_{\mathrm{cv}}(\pi^! B) \to \mathrm{H}^{\bullet - k}(B, o(E)) \tag{3.138}$$

is an isomorphism. If $\theta \in \mathrm{H}^k_{\mathrm{cv}}(E, \pi^\sharp o(E)^*)$ denotes the Thom class of the vector bundle, the inverse of $(\pi^!)_*$ is given by

$$\mathrm{H}^{\bullet - k}(B, o(E)) \ni \omega \mapsto \sharp^*_{\pi^! B}\theta \wedge (\pi^!)^*\omega \in \mathrm{H}^\bullet_{\mathrm{cv}}(\pi^! B). \tag{3.139}$$

Proof. The statement follows from a spectral sequence argument. Consider the filtration on $\Omega^\bullet_{\mathrm{cv}}(\pi^! B)$ given by

$$\mathcal{F}^p\Omega^\bullet_{\mathrm{cv}}(\pi^! B) = (\Omega_{\mathrm{cv}}(\mathcal{F}(\pi))^{\wedge p} \wedge \Omega_{\mathrm{cv}}(\pi^! B))^\bullet$$

and the induced spectral sequence $\{E^{p,q}_r\}_{r \geq 0}$. By standard arguments we obtain

$$E^{p,q}_1 \simeq \mathrm{H}^q(B, \mathrm{H}^p_{\mathrm{cv}}(\mathcal{F}(\pi))).$$

Then by Lemma 3.8.5 the statement follows. □

3.8.2 The Gysin sequence

We prove a Gysin-like sequence, which we have used in Section 3.5, for the cohomology of $\pi^! B$, where $B \Rightarrow N$ is a Lie algebroid and $\pi\colon \mathbb{S} \to N$ is a sphere bundle of rank k, i.e. a locally trivial fibre bundle with a k-dimensional sphere as typical fibre. In this case, by the Serre spectral sequence for fibre bundles [MS24, Theorem 4.11] (Theorem 2.3.11) we obtain a spectral sequence converging to the Lie algebroid cohomology of $\pi^! B$ with second page given by

$$E^{p,q}_2 = \mathrm{H}^p(B, \mathcal{H}^q(\mathbb{S})). \tag{3.140}$$

Here, $\mathcal{H}^q(\mathbb{S}) \to N$ denotes a smooth vector bundle, its fibres given by $\mathcal{H}^q(\mathbb{S})_x = \mathrm{H}^q(\pi^{-1}(x))$ for $x \in N$. Thus, $E^{p,q}_2$ has nontrivial entries only if $q = 0$ or $q = k$. Therefore, the next (and last) nontrivial differential is d_{k+1}. Recall that in case of $B = TN$ and trivial orientation bundle it is given by $\mathrm{d}_{k+1} = \wedge e$, where $e \in \mathrm{H}^{k+1}(N)$ is the Euler class of the sphere bundle [BT82, Chapter 11]. If $o(\mathbb{S})$ is nontrivial, we can consider the pullback to a trivialising double cover $\tilde{N} \to N$ and find that $e \in \mathrm{H}^{k+1}(\tilde{N})_- = \mathrm{H}^{k+1}(N, o(\mathbb{S}))$ instead.

Theorem 3.8.10 *Let $B \Rightarrow N$ be a Lie algebroid with anchor $\sharp$ and $\pi\colon \mathbb{S} \to N$ a sphere bundle of rank k. Then there is a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^\bullet(B) \xrightarrow{(\pi^!)^*} \mathrm{H}^\bullet(\pi^! B) \xrightarrow{(\pi^!)_*} \mathrm{H}^{\bullet - k}(B, o(\mathbb{S})) \xrightarrow{\wedge \sharp^* e} \mathrm{H}^{\bullet + 1}(B) \longrightarrow \ldots \tag{3.141}$$

Here, $(\pi^!)_*$ denotes fibre integration and $e \in \mathrm{H}^{k+1}(N, o(\mathbb{S}))$ is the Euler class of the sphere bundle.

Proof. If $o(\mathbb{S})$ is nontrivial, we can pull everything back to a trivialising double cover. Using Lemma 3.2.1 and noting that integration and $\wedge e$ map $\mathbb{Z}_2$-invariant classes to anti-invariant ones and vice versa, one then obtains the result in general. Thus, let $o(\mathbb{S})$ be trivial. The differential $\mathrm{d}_{k+1}\colon E_2^{0,k} \to E_2^{k+1,0}$ can be computed by evaluating on a generator s. We can write $s = \sum_i \chi_i[\omega_i]$, where $\omega_i \in \Omega^k(\mathbb{S}|_{U_i})$ are closed with respect to the de Rham differential and $\chi_i \in \mathscr{C}^\infty(N)$. Let $\gamma\colon \pi^\sharp B \to \pi^! B$ be a splitting, $\omega = \sum_i \chi_i \omega_i$, and $a_0, \dots, a_k \in \Gamma(B)$. Using that for $B = TN$ we have $\mathrm{d}_{\mathrm{dR}}\omega = \pi^* e$, by a standard calculation we obtain

$$\begin{aligned}
&\mathrm{d}_{k+1}\omega(\gamma\pi^\sharp a_0, \dots, \gamma\pi^\sharp a_k)\\
&\quad= \sum_{i=0}^{k} (-1)^i \sharp_{\pi^! A}(\gamma\pi^\sharp a_i)\omega(\gamma\pi^\sharp a_0, \dots, \overset{i}{\wedge}, \dots, \gamma\pi^\sharp a_k)\\
&\qquad + \sum_{0\le i<j\le k} (-1)^{i+j}\omega([\gamma\pi^\sharp a_i, \gamma\pi^\sharp a_j], \gamma\pi^\sharp a_0, \dots, \overset{i}{\wedge}, \dots, \overset{j}{\wedge}, \dots, \gamma\pi^\sharp a_k)\\
&\quad= \sum_{i=0}^{k} (-1)^i \sharp_{\pi^! A}(\gamma\pi^\sharp a_i)\omega(\sharp_{\pi^! A}(\gamma\pi^\sharp a_0), \dots, \overset{i}{\wedge}, \dots, \sharp_{\pi^! A}(\gamma\pi^\sharp a_k))\\
&\qquad + \sum_{0\le i<j\le k} (-1)^{i+j}\omega(\sharp_{\pi^! A}([\gamma\pi^\sharp a_i, \gamma\pi^\sharp a_j]),\\
&\qquad\qquad\qquad \sharp_{\pi^! A}(\gamma\pi^\sharp a_0), \dots, \overset{i}{\wedge}, \dots, \overset{j}{\wedge}, \dots, \sharp_{\pi^! A}(\gamma\pi^\sharp a_k))\\
&\quad= \mathrm{d}_{\mathrm{dR}}\omega(\sharp_{\pi^! A}(\gamma\pi^\sharp a_0), \dots, \sharp_{\pi^! A}(\gamma\pi^\sharp a_k))\\
&\quad= \pi^* e(\sharp_{\pi^! A}(\gamma\pi^\sharp a_0), \dots, \sharp_{\pi^! A}(\gamma\pi^\sharp a_k))\\
&\quad= \pi^*(\sharp_A^* e(a_0, \dots, a_k)).
\end{aligned}$$

The rest then follows analogously to the proof of [BT82, Proposition 14.33]. □

If $\mathbb{S} \to N$ actually comes from a vector bundle $\mathrm{pr}\colon V \to N$, we can rewrite the Gysin sequence using the Thom isomorphism on $\mathrm{H}^{\bullet-(k-1)}(B, o(\mathbb{S}))$.

Lemma 3.8.11 *Let $B \Rightarrow N$ be a Lie algebroid with anchor $\sharp$ and $\mathrm{pr}\colon V \to N$ a vector bundle of rank k. Then, under the isomorphism $\mathrm{H}^\bullet(B) = \mathrm{H}^\bullet(\mathrm{pr}^! B)$ induced by $(\iota^!)^*$, the diagram*

$$\begin{tikzcd}
\mathrm{H}^{\bullet-k}(B, o(\mathbb{S})) \arrow[r, "\sharp^* e\wedge"] \arrow[d, "\Phi"', "\cong"] & \mathrm{H}^\bullet(B) \\
\mathrm{H}^\bullet_{\mathrm{cv}}(\mathrm{pr}^! B) \arrow[r, "i"] & \mathrm{H}^\bullet(\mathrm{pr}^! B) \arrow[u, "\cong", "(\iota^!)^*"']
\end{tikzcd}$$

commutes, where $\Phi\colon \mathrm{H}^{\bullet-k}(B, o(\mathbb{S})) \to \mathrm{H}^\bullet_{\mathrm{cv}}(\mathrm{pr}^! B)$ denotes the Thom isomorphism from Lemma 3.8.9, $\iota^!\colon B \to \mathrm{pr}^! B$ is the inclusion of Lie algebroids over the zero section $\iota\colon N \to V$, and i denotes the natural map of considering a compact vertical form as just a form on $\mathrm{pr}^! B$.

Proof. First note that $o(V) = o(\mathbb{S})$ since the sphere bundle is associated to V [BT82, Proposition 11.2]. Again, we will only prove the statement for orientable vector bundles. Writing $\theta \in \mathrm{H}^k_{\mathrm{cv}}(V)$ for the Thom class, for any $\omega \in \mathrm{H}^\bullet(B)$ we have

$$(\iota^!)^*\Phi(\omega) = (\iota^!)^*\sharp^*_{\mathrm{pr}^!B}\theta \wedge \omega = (\sharp_{\mathrm{pr}^!B} \circ \iota^!)^*\theta \wedge \omega = (T\iota \circ \sharp)^*\theta \wedge \omega = \sharp^* e \wedge \omega$$

as the pullback of θ by the zero section is the Euler class. □

Corollary 3.8.12 *Let* $B \Rightarrow N$ *be a Lie algebroid, and* $\mathrm{pr}\colon V \to N$ *a vector bundle with induced sphere bundle* $\pi\colon \mathbb{S} \to N$. *Then there is a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^\bullet(B) \xrightarrow{(\pi^!)^*} \mathrm{H}^\bullet(\pi^! B) \xrightarrow{(\pi^!)_*} \mathrm{H}^{\bullet+1}_{\mathrm{cv}}(\mathrm{pr}^! B) \xrightarrow{i} \mathrm{H}^{\bullet+1}(B) \longrightarrow \ldots \quad (3.142)$$

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Chapter 4

Blowups of Dirac structures

This chapter contains [MSZ25], submitted. I did not contribute to Section 4.8.

Abstract

Given a real, twisted Dirac structure D on a smooth manifold M, and a closed embedded submanifold $N \subseteq M$ of codimension > 1, we characterise when D lifts to a smooth, twisted Dirac structure on the real projective blowup of M along N. This holds precisely when N is either a submanifold transverse to D (with no further restrictions) or a submanifold invariant for D, for which the Lie algebras transverse to N have all of the same constant height $k \geq 0$. In this paper, we also classify Lie algebras satisfying this Lie-theoretic property up to isomorphism: Lie algebras of constant height $k = 0$ are either abelian or a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^n$ for the diagonal representation of $\mathbb{R}$ on $\mathbb{R}^n$; there is only $\mathfrak{so}(3)$ of constant height $k = 1$; there are no Lie algebras of constant height $k \geq 2$. We recover a theorem of Polishchuk, which establishes that a Poisson structure lifts to a Poisson structure on the blowup of a submanifold exactly when the submanifold is invariant and the transverse Lie algebras have constant height $k = 0$.

4.1 Introduction

Given a manifold M and a closed and embedded submanifold N, the **real projective blowup** is obtained by replacing N by $\mathbb{P}(\nu_N(M))$, the projectivisation of the normal bundle $\nu_N(M) \to N$ of N in M. This yields a smooth manifold (without boundary), which we denote by $\mathrm{Blup}(M, N)$. The blowup comes with a canonical blowdown map

$$p\colon \mathrm{Blup}(M, N) \to M,$$

which on $\mathbb{P}(\nu_N(M)) = p^{-1}(N)$ is the natural projection to N, and which defines a diffeomorphism from the complement of $\mathbb{P}(\nu_N(M))$ to the complement of N. In particular, p is not a submersion.

One can address the question of when geometric structures on M "lift" to the blowup. This question has been addressed in many contexts in the literature, both to produce new examples and as a way to desingularise geometric structures.

Notice that we are considering real blowups, and that in the literature complex blowups also appear frequently.

In [Pol97], Polishchuk studied the projective blowup of Poisson schemes, i.e. the existence of a Poisson structure on the blowup such that the blowdown map is Poisson. We state the result of [Pol97, Section 8] in its weaker, differential-geometric formulation, in a similar way to [GL13, Section 2.2] (see Chapter 5 for a detailed proof).

Recall that, given a Poisson submanifold N of a Poisson manifold (M, π), the conormal space at any $q \in N$,

$$(T_qN)^{\mathrm{ann}} = (T_qM/T_qN)^*$$

carries a Lie bracket, turning $(TN)^{\mathrm{ann}}$ into a bundle of Lie algebras. Explicitly, the bracket is given by

$$[(\mathrm{d}f)(q), (\mathrm{d}g)(q)] := (\mathrm{d}\{f, g\})(q), \tag{4.1}$$

where f, g are smooth functions on M vanishing on N.

Theorem (Polishchuk [Pol97]) *Let (M, π) be a Poisson manifold and $N \subseteq M$ a closed and embedded Poisson submanifold. There exists a Poisson bivector field $\tilde{\pi}$ on $\mathrm{Blup}(M, N)$ which is p-related to π if and only if each Lie algebra $(T_yN)^{\mathrm{ann}}$, for $y \in N$, is either abelian or isomorphic to the semi-direct product by the diagonal representation $\lambda \mapsto \lambda \mathrm{id}_{\mathbb{R}^n}$ of $\mathbb{R}$ on $\mathbb{R}^n$, denoted by $\mathbb{R} \ltimes \mathbb{R}^n$.*

In this paper we shall be interested in lifting **twisted Dirac structures**. Dirac structures include Poisson structures, closed 2-forms, and foliations as special cases, and are suitable to describe geometric structures that become "infinite" at certain points.

Given a (twisted) Dirac structure $D \subseteq TM \oplus T^*M$ on a manifold M and a closed, embedded submanifold $N \subseteq M$ with $\operatorname{codim}(N) > 1$, we ask:

Does there exist a (twisted) Dirac structure $\tilde{D}$ on $\mathrm{Blup}(M, N)$, such that

$$\tilde{D}\big|_{\mathrm{Blup}(M,N)\setminus\mathbb{P}(\nu_N(M))} = p^*(D\big|_{M\setminus N})?$$

If such a Dirac structure $\tilde{D}$ exists, it is necessarily unique. In this case, we say that D **lifts to the blowup**. Further, we ask whether **the blowdown map is forward or backward Dirac**.

Note that for $\operatorname{codim}(N) = 1$ there is nothing to show, as $\mathrm{Blup}(M, N) \simeq M$ via the blowdown map and any Dirac structure lifts.

Our main theorem is an extension of Polishchuk's result (in the smooth setting, formulated before) to twisted Dirac structures. To state it, first notice that, given a twisted Dirac structure D on M and an invariant submanifold $N \subseteq M$, the annihilator $(TN)^{\mathrm{ann}}$ is a bundle of Lie algebras, as in the case of Poisson structures. Indeed, $(TN)^{\mathrm{ann}}$ fits into a short exact sequence of Lie algebroids:

$$0 \longrightarrow (TN)^{\mathrm{ann}} \longrightarrow D\big|_N \longrightarrow \mathfrak{B}_{\iota_N} D \longrightarrow 0,$$

where $D\big|_N$ denotes the pullback of the Lie algebroid D to N and $\mathfrak{B}_{\iota_N} D$ denotes the pullback of the twisted Dirac structure D to N.

Main Theorem *Let D be a twisted Dirac structure on a manifold M and $N \subseteq M$ a connected, closed, and embedded submanifold of codimension* $\operatorname{codim} N > 1$. *Then D lifts to a twisted Dirac structure on* $\mathrm{Blup}(M, N)$ *if and only if one of the following conditions holds.*

(1) $N \subseteq M$ is a transversal.

(2) $N \subseteq M$ is an invariant submanifold and the induced bundle of Lie algebras $(TN)^{\mathrm{ann}} \to N$ satisfies one of the following conditions:

  (a) Each fibre is either abelian or isomorphic to the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^n$ by the diagonal representation of $\mathbb{R}$ on $\mathbb{R}^n$.

  (b) The fibres are all isomorphic to $\mathfrak{so}(3)$.

The blowdown map is a backward Dirac map in case (1), and it is a forward Dirac map in case (2).

Remark If N is a presymplectic leaf of the Dirac structure D, or equivalently, if N is invariant and $D\big|_N$ is a transitive Lie algebroid, then $(TN)^{\mathrm{ann}}$ is a locally trivial bundle of Lie algebras. Hence all fibres are isomorphic to each other, simplifying the statement of the main theorem.

Example *As a simple illustration of the above theorem, let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a proper ideal. Then $\mathfrak{h}^{\mathrm{ann}}$ is a Poisson submanifold of $\mathfrak{g}^*$, endowed with the standard linear Poisson structure. The latter lifts to a Dirac structure on* $\mathrm{Blup}(\mathfrak{g}^*, \mathfrak{h}^{\mathrm{ann}})$ *if and only if the Lie algebra $\mathfrak{h}$ is isomorphic to one of the Lie algebras appearing in item (2) of the main theorem.*

The outline of the paper and intermediate results. Sections 4.2 and 4.3 are devoted to recalling basic notions and results about the blowup construction and Dirac structures, respectively.

The proof of the Main Theorem is divided into several steps corresponding to the remaining sections of the paper.

The case of transverse submanifolds is easily dealt with in Section 4.4, where the following is proven.

Theorem (4.4.2) *Let D be a twisted Dirac structure on M. For any closed, embedded transverse submanifold $N \subseteq M$, D lifts to* $\mathrm{Blup}(M, N)$ *and the blowdown map is a backward Dirac map.*

In Section 4.5 we prove the transverse or invariant dichotomy from the Main Theorem.

Theorem (4.5.1) *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed, embedded, and connected submanifold of codimension > 1. If D lifts to a twisted Dirac structure on* $\mathrm{Blup}(M, N)$, *then one of the two conditions holds.*

- *$N \subseteq M$ is a transversal. In this case, the blowdown map is a backward Dirac map.*
- *$N \subseteq M$ is an invariant submanifold. In this case, the blowdown map is a forward Dirac map.*

To prove this result, we make use of the description of Dirac structures in terms of spinor lines, i.e. line subbundles of $\oplus_k \wedge^k T^*M$ [Gua11]. For this, we develop in Subsection 4.5.1 techniques to study the problem of extending line bundles along codimension one submanifolds—some of the obtained results are similar those in [Blo17]. In particular, our techniques imply the following result.

Corollary (4.5.6) *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed and embedded submanifold. There exists an open and dense subset $V \subseteq \mathbb{P}(\nu_N(M))$ such that the Dirac structure $p^*(D\big|_{M\setminus N})$ extends smoothly to a Dirac structure on $\big(\mathrm{Blup}(M,N) \setminus \mathbb{P}(\nu_N(M))\big) \cup V$.*

In Section 4.6 we turn to the general problem of blowing up invariant submanifolds.

The tools developed in Section 4.5 are crucial also here. Namely, we use spinors to describe Dirac structures, and study the vanishing order along the divisor of the pullback of the spinor to the blowup.

Our goal is to reduce the problem to the following purely Lie theoretical property of the isotropy Lie algebras $(T_qN)^{\mathrm{ann}}$, $q \in N$. For a Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$, define the **height** of an element $\xi \in \mathfrak{g}^* \setminus \{0\}$ as the integer $k \in \mathbb{N}_0$ satisfying

$$\xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^k \neq 0 \quad \text{and} \quad \xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} = 0,$$

where $\mathrm{d}_{\mathfrak{g}}$ denotes the Chevalley-Eilenberg differential. We say that $\mathfrak{g}$ is a Lie algebra of **constant height** $k \geq 0$, if all elements $\xi \in \mathfrak{g}^* \setminus \{0\}$ have height k.

Next, we note that the structure of a bundle of Lie algebras of $(TN)^{\mathrm{ann}}$ induces a fibrewise linear Poisson structure on the normal bundle $\nu_N(M) \simeq ((TN)^{\mathrm{ann}})^*$, which we denote by π_{lin}. The Dirac structure $\mathrm{graph}(\pi_{\mathrm{lin}})$ can be thought of as the linearisation of the vertical component of D at N.

The main results of Section 4.6 are summarised in the following.

Theorem (4.6.1, 4.6.3, 4.6.6, 4.6.7) *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed, embedded and connected submanifold, which is invariant. The following are equivalent.*

- *D lifts to $\mathrm{Blup}(M,N)$.*
- *There exists $\ell \in \{1, \dots, \mathrm{codim}\, N - 1\}$ such that, for any (local) spinor ϕ for D, $p^*\phi$ has constant vanishing order ℓ along $\mathbb{P}(\nu_N(M))$.*
- *$\mathrm{graph}(\pi_{\mathrm{lin}})$ lifts to $\mathrm{Blup}(\nu_N(M), 0_N)$.*
- *The Lie algebras $(T_qN)^{\mathrm{ann}}$, $q \in N$, have all the same constant height k.*

Finally, using structure theory of semisimple Lie algebras, in Section 4.8 we classify Lie algebras of constant height.

Theorem (4.8.1) *Any Lie algebra* $\mathfrak{g}$ *of constant height is isomorphic to one of the following.*

- *An abelian Lie algebra* $\mathbb{R}^n$*—this has height* 0.
- *The semi-direct product* $\mathbb{R} \ltimes \mathbb{R}^n$*, for the representation* $\lambda \mapsto \lambda \mathrm{id}_{\mathbb{R}^n}$*—this has height* 0.
- *The Lie algebra* $\mathfrak{so}(3)$*—this has height* 1.

Theorems 4.4.2, 4.5.1, 4.6.1 and 4.8.1 together yield the Main Theorem.

In Section 4.7, for the special case of blowing up a zero of a Poisson structure, we provide an alternative, more geometric proof of Theorem 4.6.1, without using spinors, independently of our previous arguments. We do so in Theorem 4.7.1, which gives several geometric characterisations of the existence of lifts, one of them in terms of the dimensions of the coadjoint orbits.

Relation to the literature

The main question we address in this paper is motivated by Polishchuk's result [Pol97, Section 8] recalled above. In particular, our Main Theorem extends Polishchuk's result in the framework of smooth manifolds from Poisson to Dirac structures. Our techniques differ from those used by Polishchuk in [Pol97, Section 8]. There, Poisson structures are described by means of the Poisson bracket on the algebra of functions; since for Dirac structures only the subalgebra of admissible functions is endowed with a Poisson bracket, we use more geometric techniques. It is however surprising that, although we allow for completely general submanifolds and completely general twisted Dirac structures, we obtain only a few new situations compared with Polishchuk's setting in which the Dirac structure lifts to the blow-up: transverse submanifolds and invariant submanifolds of codimension three with transverse Poisson structure $\mathfrak{so}(3)$.

Indirectly, the case of $\mathfrak{so}(3)$ has appeared in the literature in relation to the problem of linearising a Poisson structure π around a zero q (i.e. $\pi(q) = 0$). Namely, if the isotropy Lie algebra of π at q is $\mathfrak{so}(3)$, a geometric proof of Conn's linearisation theorem [Con85] can be obtained by first blowing up q, then using Reeb's stability for the regular foliation underlying the lifted Dirac structure, and finally using a foliated Moser trick. This proof, envisioned by Weinstein [Wei83], was carried out by Dazord in [Daz85], and in fact, using the language of completely integrable 1-forms, the argument can be traced back to Reeb's thesis [Ree52].

Blowups have also been studied in the setting of generalised complex geometry in [CG09] in dimension 4 and in [BCvdLD19] in greater generality. The natural operation in this setting is the complex blowup, arising from a complex structure on the normal bundle to the submanifold. Like a foreshadowing of the "transverse-invariant-dichotomy" revealed in our paper, the submanifolds considered in [BCvdLD19] are certain classes of Poisson submanifolds for which Polishchuk's result applies and certain Poisson transversals, with a transverse complex structure. In the second case, in order to obtain a generalised complex structure on the blowup, the lifted (complex) Dirac structure needs to be modified around the divisor using an adaptation of blowup construction in symplectic geometry [GS89] and Lermann's symplectic cut construction [Ler95] to generalized complex geometry.

It would be interesting to see if there is more flexibility in the more general setting of weighted blowups, which have recently proven to be very successful in simplifying Hironaka's resolutions of singularities in algebraic geometry [ATW24, McQ20]. For Poisson structures, this is currently being investigated by Pym and collaborators [LMP].

Every Dirac manifold (M, D) carries a natural singular foliation $\mathcal{F} = \mathrm{pr}_{TM}(\Gamma_c(D))$, whose leaves are precisely the presymplectic leaves. For an invariant (respectively transverse) submanifold $N \subseteq M$, the singular foliation $\mathcal{F}$ lifts to a singular foliation $p^{-1}\mathcal{F}$ on the blowup $\mathrm{Blup}(M, N)$—see [LGLR24, Section 1.5.7] (respectively [LGLR24, Section 1.5.4]). In the case when the Dirac structure also lifts, one has the inclusion $p^{-1}\mathcal{F} \subseteq \widehat{\mathcal{F}}$ into the singular foliation $\widehat{\mathcal{F}}$ of the lifted Dirac structure, which in general is strict. When N is a transverse submanifold, the above inclusion is an equality [SZ25].

Crainic, Fernandes, and Martínez Torres [CFT25] develop a procedure to lift a Poisson structure of compact type on a manifold M to a regular Dirac structure on another manifold $\widehat{M}$ of the same dimension. However, $\widehat{M}$ is in general not a real projective blowup. In the special case of the Poisson manifold $\mathfrak{so}(3)^*$, their construction does agree with the real projective blowup, and fits in the setting of our Main Theorem and of Theorem 4.7.1.

Finally, Theorem 4.6.1 classifies (real) Lie algebras of constant height. We believe that this statement and its proof are of independent interest. In Remark 4.8.3 we relate the height of an element with the Cartan class of that element, as used for instance in [GR19] to obtain classification results.

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4.2 Real projective blowups

In this section, we briefly review the real projective blowup and some of its basic properties needed for this chapter.

As a set, the blowup of a closed, embedded submanifold N of a manifold M is given by

$$\mathrm{Blup}(M, N) = (M \setminus N) \sqcup \mathbb{P}(\nu_N(M)),$$

i.e. N is replaced with the projectivisation of the normal bundle $\nu_N(M) := TM\big|_N/TN$ of N in M. The blowup comes with a canonical smooth structure, for which the blowdown map,

$$p\colon \mathrm{Blup}(M,N) \to M,$$

is smooth. The blowdown map is the identity on $M \setminus N$ and the projection on $\mathbb{P}(\nu_N(M)) \to N$. If $\operatorname{codim} N = 1$, then p is a diffeomorphism. The smooth structure can be described via the following set of charts.

Charts for $\mathrm{Blup}(M,N)$

Let $(U,(x,y))$ be a submanifold chart for N, i.e. $U \cap N = \{x = 0\}$. Then the collection

$$U_i = p^{-1}\big(\{x_i \neq 0\}\big) \cup \big\{[v] \in \mathbb{P}(\nu_N(M))\big|_{U\cap N}) : \mathrm{d}x_i(v) \neq 0\big\},$$

$i = 1,\ldots,\operatorname{codim}(N)$, is an open cover of $p^{-1}(U)$. On each U_i, one defines coordinates $(\tilde{x},y)$ on $\mathrm{Blup}(M,N)$, in which the blowdown map reads (see e.g. [Obs21, Remark 5.29] for details)

$$p(\tilde{x}_1,\ldots,\tilde{x}_i,\ldots,\tilde{x}_{\operatorname{codim}(N)},y) = (\tilde{x}_i\tilde{x}_1,\ldots,\tilde{x}_i,\ldots,\tilde{x}_i\tilde{x}_{\operatorname{codim}(N)},y). \tag{4.2}$$

Notice that the hyperplane $\mathbb{P}(\nu_N(M)) \cap U_i$ is given by $\tilde{x}_i = 0$, and that the chart obtained by restriction is the well-known chart on the projective bundle induced by the fibrewise linear coordinates $(\mathrm{d}x, y)$ on the normal bundle $\nu_N(M)$.

Lifts of vector fields to $\mathrm{Blup}(M,N)$

In the chart $(U_i,(\tilde{x},y))$, the lifts of the coordinate vector fields $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_k}$ for $k \neq i$ are given by

$$\frac{\partial}{\partial \tilde{x}_i} - \sum_{k\neq i} \frac{\tilde{x}_k}{\tilde{x}_i}\frac{\partial}{\partial \tilde{x}_k} \qquad \text{and} \qquad \frac{1}{\tilde{x}_i}\frac{\partial}{\partial \tilde{x}_k}, \tag{4.3}$$

respectively, as one sees by applying these vector fields to functions of the form p^*x_j. This implies the following standard result (see e.g. [LGLR24, Prop. 1.5.40] for points or [Sch24, Lemma 3.5]).

Lemma 4.2.1 *A vector field X on M is p-related to some vector field $\tilde{X}$ on* $\mathrm{Blup}(M,N)$ *if and only if X is tangent to N. In that case, $\tilde{X}$ is unique and tangent to* $\mathbb{P}(\nu_N(M)) \subseteq \mathrm{Blup}(M,N)$.

4.3 Dirac structures

We give a short introduction to Dirac structures by first describing them as subbundles of the standard Courant algebroid (see Section 1.1.3) and then via their spinor lines. Dirac structures were introduced in [Cou90, CW88] and we recommend [Bur13] for a brief introduction to the field, and [Gua11] for the use of spinors.

4.3.1 Dirac structures as Lagrangian subbundles of $\mathbb{T}M$

Let M be a manifold and $H \in \Omega^3(M)$ a closed 3-form. The **generalised tangent bundle**

$$\mathbb{T}M := TM \oplus T^*M$$

carries a **nondegenerate pairing** and the **Dorfman bracket**, defined by

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle \colon \mathbb{T}M \times \mathbb{T}M &\to \mathbb{R} \\ \begin{pmatrix} v \\ \xi \end{pmatrix}, \begin{pmatrix} w \\ \eta \end{pmatrix} &\mapsto \frac{1}{2}(\xi(w) + \eta(v)), \\ [\,\cdot\,,\,\cdot\,] \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) &\to \Gamma(\mathbb{T}M) \\ \begin{pmatrix} X \\ \alpha \end{pmatrix}, \begin{pmatrix} Y \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} [X, Y] \\ \mathscr{L}_X \beta - \mathrm{i}_Y \mathrm{d}\alpha + \mathrm{i}_Y \mathrm{i}_X H \end{pmatrix}. \end{aligned}$$

Together with the **anchor** map $\sharp = \mathrm{pr}_{TM} \colon \mathbb{T}M \to TM$ this endows $\mathbb{T}M$ with the structure of a H**-twisted Courant algebroid**.

Definition 4.3.1 A subbundle $D \subseteq \mathbb{T}M$ is called an H**-twisted Dirac structure** on M if

1. D is maximally isotropic (i.e. **Lagrangian**) with respect to the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$,
2. D is involutive, i.e. $[\Gamma(D), \Gamma(D)] \subseteq \Gamma(D)$.

When we don't want to specify H, we call D a simply a **twisted Dirac structure**, or even simply **a Dirac structure** (even if $H \neq 0$).

Here are the standard examples for H-twisted Dirac structures, following [Gua11, Example 2.11-2.13].

1. The graph of a bivector field π on M,
$$\mathrm{graph}(\pi) = \left\{ \begin{pmatrix} \pi^\sharp \xi \\ \xi \end{pmatrix} \in \mathbb{T}M : \xi \in T^*M \right\},$$
if and only if
$$[\![\pi, \pi]\!]_{\mathrm{S}} = 2(\wedge^3 \pi^\sharp) H,$$
where $[\![\,\cdot\,,\,\cdot\,]\!]_{\mathrm{S}}$ is the Schouten bracket. Then π is called an H**-twisted Poisson structure** [ŠW01].
2. The graph of a 2-form ω on M,
$$\mathrm{graph}(\omega) = \left\{ \begin{pmatrix} v \\ \omega^\flat v \end{pmatrix} \in \mathbb{T}M : v \in TM \right\},$$
if and only if $\mathrm{d}\omega = H$.
3. The subbundle $T\mathcal{F} \oplus (T\mathcal{F})^{\mathrm{ann}}$, for a regular foliation $\mathcal{F}$ on M, if and only if $H\big|_{T\mathcal{F}} = 0$. Here, $\cdot^{\,\mathrm{ann}}$ denotes the annihilator.

Definition 4.3.2 ([BR03, Wei82]) Let $\phi\colon X \to M$ be a smooth map. Let D_X be an H_X-twisted Dirac structure on X, and D_M be an H_M-twisted Dirac structure on M. If $H_X = \phi^* H_M$, we call ϕ

1. **a forward Dirac map**, if we have $(D_M)_{\phi(x)} = \mathfrak{F}_\phi((D_X)_x)$ for all $x \in X$, where
$$\mathfrak{F}_\phi((D_X)_x) := \left\{ \begin{pmatrix} (T_x\phi)w \\ \xi \end{pmatrix} \in \mathbb{T}_{\phi(x)}M : \begin{pmatrix} w \\ (T_x\phi)^*\xi \end{pmatrix} \in (D_X)_x \right\};$$
2. **a backward Dirac map**, if we have $(D_X)_x = \mathfrak{B}_\phi((D_M)_{\phi(x)})$ for all $x \in X$, where
$$\mathfrak{B}_\phi((D_M)_{\phi(x)}) = \left\{ \begin{pmatrix} w \\ (T_x\phi)^*\xi \end{pmatrix} \in \mathbb{T}_x X : \begin{pmatrix} (T_x\phi)w \\ \xi \end{pmatrix} \in (D_M)_{\phi(x)} \right\}.$$

Forward Dirac maps generalise Poisson maps, while backward maps generalise pullbacks of 2-forms.

The **gauge action** of a 2-form $B \in \Omega^2(M)$ on a subbundle $D \subseteq \mathbb{T}M$ is defined by [ŠW01]
$$\exp^B D := \left\{ \begin{pmatrix} X \\ \alpha + \mathrm{i}_X B \end{pmatrix} \in \mathbb{T}M : \begin{pmatrix} X \\ \alpha \end{pmatrix} \in D \right\}.$$
Gauge actions send Lagrangian subbundles to Lagrangian subbundles. Moreover, $D \subseteq \mathbb{T}M$ is an H-twisted Dirac structure if and only if $\exp^B D \subseteq \mathbb{T}M$ is an $(H + \mathrm{d}B)$-twisted Dirac structure.

Definition 4.3.3 For $i = 1, 2$, let D_i be an H_i-twisted Dirac structure on M_i. We say that D_1 and D_2 are **isomorphic** if there exists a diffeomorphism $f\colon M_1 \to M_2$ and a 2-form $B_2 \in \Omega^2(M_2)$ such that
$$D_1 = f^*(\exp^{B_2} D_2) \quad \text{and} \quad H_1 = f^*(H_2 + \mathrm{d}B_2).$$
We call D_1 and D_2 **locally isomorphic** around $x_1 \in M_1$ and $x_2 \in M_2$ if their restrictions to open neighbourhoods U_1 of x_1 and U_2 of x_2, respectively, are isomorphic.

4.3.2 Spinor description of Dirac structures

Another way to describe Dirac structures is by means of pure spinor lines. Let M be a manifold. Define an action ρ of $\Gamma(\mathbb{T}M)$ on $\Omega^\bullet(M)$ by
$$\rho \begin{pmatrix} X \\ \alpha \end{pmatrix} \phi := \mathrm{i}_X \phi + \alpha \wedge \phi.$$
Then, for a nowhere vanishing $\phi \in \Omega^\bullet(M)$, called a **spinor**, the spaces
$$D_\phi := \left\{ \begin{pmatrix} X \\ \alpha \end{pmatrix} \in \mathbb{T}M : \rho \begin{pmatrix} X \\ \alpha \end{pmatrix} \phi = 0 \right\}$$

are isotropic in every point of M, i.e. $D_\phi \subseteq D_\phi^\perp$, and ϕ is called a **pure spinor** if these spaces are of maximal dimension, i.e. $\operatorname{rank} D_\phi = \dim M$. The maximal isotropic subbundle corresponding to a pure spinor depends only on the line bundle

$$\Sigma := \mathbb{R} \cdot \phi \subseteq \wedge^\bullet T^*M.$$

Conversely, every maximal isotropic subbundle has a corresponding pure spinor line bundle ([Che97, III.1.9], [Gua11, Proposition 1.3]). Further, D_ϕ is an H-twisted Dirac structure if and only if the spinor ϕ satisfies the following condition [Gua11, Theorem 2.9]: there exists $A \in \Gamma(\mathbb{T}M)$ such that

$$\mathrm{d}_H \phi = \rho(A)\phi, \quad \text{where } \mathrm{d}_H := \mathrm{d} - H \wedge .$$

Example 4.3.4

1. A spinor of $TM \subseteq \mathbb{T}M$ is given by the constant function $1 \in \Omega^\bullet(M)$, and the spinor line is $\mathbb{R} \subseteq \wedge^\bullet T^*M$.

2. A local spinor of $T^*M \subseteq \mathbb{T}M$ is given by any local volume form on M, and the spinor line is $\wedge^{\mathrm{top}} T^*M \subseteq \wedge^\bullet T^*M$.

3. Let (M, π) be twisted Poisson. On any orientable open subset of M with volume form λ a spinor for $\operatorname{graph}(\pi)$ is given by

$$\phi_\pi = \exp^{\mathrm{i}_\pi} \lambda = \sum_{k=0} \frac{1}{k!} \mathrm{i}_\pi^k \lambda.$$

   Here, we adopt the convention $\mathrm{i}_{X \wedge Y} = \mathrm{i}_Y \mathrm{i}_X$ for the insertion of multivector fields. Further, any spinor for $\operatorname{graph}(\pi)$ over such an open subset has this form, for some volume form λ. In particular, the top-degree component of the spinor is nowhere vanishing.

4. Let $\omega \in \Omega^2(M)$ be a 2-form. A spinor for $\operatorname{graph}(\omega)$ is given by

$$\phi_\omega = \exp^\omega = \sum_{k=0} \frac{\omega^{\wedge k}}{k!}.$$

We give a useful criterion for extending Dirac structures (for similar results, see [Blo17]).

Lemma 4.3.5 *Let M be a smooth manifold, H a closed 3-form on M, and $U \subseteq M$ an open and dense subset. Let D be an $H\big|_U$-twisted Dirac structure on U, with spinor line $\Sigma \subseteq \wedge^\bullet T^*U$. The following are equivalent.*

- *D extends to an H-twisted Dirac structure $\tilde{D}$ on U;*
- *Σ extends to a line subbundle $\tilde{\Sigma} \subseteq \wedge^\bullet T^*U$ on U.*

Proof. Clearly, if D extends, then so does Σ. Conversely, assume that Σ extends to $\tilde{\Sigma}$. We claim that any (local) non-vanishing section ϕ of $\tilde{\Sigma}$ is a pure spinor. Indeed, D_ϕ is the kernel of a vector bundle map, so its dimension is locally non-increasing, and, at the same time, D_ϕ is almost everywhere maximally isotropic. This shows that the annihilator $\tilde{D}$ of $\tilde{\Sigma}$ is a Lagrangian extension of the twisted Dirac structure D to M. Now, $\tilde{D}$ is also twisted Dirac, because this condition is closed. ☐

4.3.3 Splitting theorems for Dirac structures

We briefly discuss invariant and transverse submanifolds of Dirac structures, and their normal forms, which play a significant role throughout the paper.

Definition 4.3.6 Let $D \subseteq \mathbb{T}M$ be a Dirac structure and $N \subseteq M$ a closed and embedded submanifold.

- N is called **invariant** if $\mathrm{pr}_{TM}(D)$ is tangent to N, i.e. for all $q \in N$

$$\mathrm{pr}_{TM}(D_q) \subseteq T_q N.$$

- N is called **transversal** if $\mathrm{pr}_{TM}(D)$ is tranverse to N, i.e. for all $q \in N$

$$T_q N + \mathrm{pr}_{TM}(D_q) = T_q M.$$

Geometrically, invariant submanifolds are those that are unions of presymplectic leaves and transverse ones are those that intersect leaves transversally.

To find local spinors we use Blohmann's normal form theorem developed in [Blo17, Corollary 3.9]. Recall that isomorphisms of twisted Dirac structure include gauge transformations by 2-forms, which are not necessarily closed (Definition 4.3.3). Therefore, any Dirac structure can be locally untwisted, and so, Blohmann's splitting theorem can be restated as follows (see also [BLM16, Theorem 5.1]).

Theorem 4.3.7 ([Blo17]) *Let D be an H-twisted Dirac structure on M and let $q \in M$. Then (M, D) is locally isomorphic around q to the (untwisted) product $(U, \mathrm{graph}(\pi)) \times (Z, TZ)$ around (q_U, q_Z), where π a Poisson bivector on U that vanishes at q_U.*

The following observation will be used later on.

Corollary 4.3.8 *In the setting of Theorem 4.3.7, let $N \subseteq M$ be an invariant submanifold through q. Then we can take $U = X \times Y$, with $q_U = (q_X, q_Y)$, so that N corresponds to $\{q_X\} \times Y \times Z$, where $\{q_X\} \times Y$ is an invariant submanifold of U.*

Proof. If Z is connected, invariant submanifolds of $\mathrm{graph}(\pi) \times TZ$ are of the form $Y \times Z$, where Y is an invariant submanifold of $(U, \mathrm{graph}(\pi))$. After shrinking U, we can choose a complement X to Y. $\square$

4.3.4 Invariance and locality of liftability

In the proof of the Main Theorem, we will frequently rely on the local description of twisted Dirac structures, as recalled in the previous subsection. This description is valid up to isomorphisms of Dirac structures (Definition 4.3.3). In this subsection, we aim to justify the use of this approach. On the one hand, liftability is a local property that is preserved under isomorphisms; on the other hand, the assumptions of the Main Theorem are local in nature and remain unaffected by isomorphisms. We formulate these observations as two lemmas.

Lemma 4.3.9 (Invariance) *For $i = 1, 2$, let D_i be an H_i-twisted Dirac structure on M_i. Assume that the two Dirac structures are isomorphic. Fix an isomorphism (f, B_2) consisting of a diffeomorphism $f\colon M_1 \xrightarrow{\sim} M_2$ and a 2-form $B_2 \in \Omega^2(M_2)$ such that*

$$D_1 = f^*(\exp^{B_2} D_2) \qquad \textit{and} \qquad H_1 = f^*(H_2 + \mathrm{d}B_2).$$

Let $N_1 \subseteq M_1$ be a closed and embedded submanifold, and denote $N_2 := f(N_1)$. The following hold.

1. *D_1 lifts to* $\mathrm{Blup}(M_1, N_1)$ *if and only if D_2 lifts to* $\mathrm{Blup}(M_2, N_2)$*;*
2. *N_1 is a transversal for D_1 if and only if N_2 is a transversal for D_2;*
3. *N_1 is an invariant submanifold for D_1 if and only if N_2 is an invariant submanifold for D_2. Moreover, in this case f induces an isomorphism of bundle of Lie algebras:*

$$(Tf^{-1})^*\colon (TN_1)^{\mathrm{ann}} \xrightarrow{\sim} (TN_2)^{\mathrm{ann}}. \tag{4.4}$$

Proof. For property 1., note that f lifts to a diffeomorphism between blowups $\hat{f}\colon \mathrm{Blup}(M_1, N_1) \xrightarrow{\sim} \mathrm{Blup}(M_2, N_2)$. Let $p_i\colon \mathrm{Blup}(M_i, N_i) \to M_i$ denote the blowdown map. Assume that D_1 lifts to a $p_1^* H_1$-twisted Dirac structure $\tilde{D}_1$ on $\mathrm{Blup}(M_1, N_1)$. This can be pushed forward via $\hat{f}$ to the Dirac structure $\hat{f}_*(\tilde{D}_1)$ on $\mathrm{Blup}(M_2, N_2)$, which is twisted by

$$\hat{f}_*(p_1^* H_1) = \hat{f}_* p_1^* f^*(H_2 + \mathrm{d}B_2) = p_2^*(H_2 + \mathrm{d}B_2).$$

Then $\tilde{D}_2 := e^{-p_2^* B_2} \circ \hat{f}_*(\tilde{D}_1)$ is a $p_2^* H_2$-twisted Dirac structure which lifts D_2. The symmetry of the statement yields the other implication.

For the other properties, note that (f, B_2) induces an isomorphism of Lie algebroids

$$e^{-B_2} \circ \mathbb{T}f\colon D_1 \xrightarrow{\sim} D_2, \qquad \begin{pmatrix} v \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} Tf(v) \\ (Tf^{-1})^*(\xi) - \mathrm{i}_{Tf(v)} B_2 \end{pmatrix},$$

and that Lie algebroid isomorphisms send transverse (respectively invariant) submanifolds to transverse (respectively invariant) submanifolds. Finally, if N_1 is invariant, then $(TN_1)^{\mathrm{ann}} \subseteq D_1$ is a subalgebroid with zero anchor (which induces the structure of bundle of Lie algebras), and the restriction of the Lie algebroid isomorphism to this subbundle is precisely (4.4). □

The properties above are also local in nature. We state this, but omit the obvious proof.

Lemma 4.3.10 (Locality) *Let D be an H-twisted Dirac structure on M and $N \subseteq M$ a closed and embedded submanifold. The following hold.*

1. *D lifts to* $\mathrm{Blup}(M, N)$ *if and only if every point in N has an open neighbourhood U in M such that $D\big|_U$ lifts to* $\mathrm{Blup}(U, N \cap U)$*;*

2. *N is a transversal for D if and only if every point in N has an open neighbourhood U in M such that $N \cap U$ is a transversal for $D\big|_U$;*
3. *N is an invariant submanifold for D if and only if every point in N has an open neighbourhood U in M such that $N \cap U$ is an invariant submanifold for $D\big|_U$.*

4.4 Blowup of transversals

In this section, we show that any Dirac structure lifts to the blowup along a transversal.

We will use the following terminology. A smooth map $f\colon B \to M$ is said to be **transverse** to a Dirac structure D on M, if, for all $b \in B$, the following holds:

$$\mathrm{pr}_{TM}(D_{f(b)}) + \mathrm{im}\, T_b f = T_{f(b)} M.$$

Geometrically, this means that f is transverse to all leaves of D.

The following characterisation and property of transverse maps are well-known (see e.g. [ABM09, Lemma 1.9] and [Bur13, Proposition 5.6]). For the convenience of the reader, a short proof is included.

Lemma 4.4.1 *Let D be a twisted Dirac structure on M, and $f\colon B \to M$ a smooth map. The following are equivalent:*

- *f is transverse to D;*
- *for any (local) spinor ϕ defining D, $f^*\phi$ is nowhere zero.*

In this case, the pullback $\mathfrak{B}_f D$ is a smooth Dirac structure defined by the (local) spinor $f^*\phi$.

Proof. For $b \in B$ denote $V_b := \mathrm{pr}_{TM}(D_{f(b)})$. Any spinor for $D_{f(b)}$ can be written as $\phi_{f(b)} = \exp^B \theta_1 \wedge \ldots \wedge \theta_n$, where $B \in \wedge^2 T^*_{f(b)} M$ and $\theta_1, \ldots, \theta_n$ are a basis of V_b^{ann}, the annihilator of V_b [Gua11, Proposition 1.3]. Then

$$(T_b f)^* \phi_{f(b)} = \exp^{(T_b f)^* B} (T_b f)^*(\theta_1) \wedge \ldots \wedge (T_b f)^*(\theta_n)$$

is non-zero precisely when $(T_b f)^*\colon V^{\mathrm{ann}} \to T_b^* B$ is injective, which is equivalent to the map $T_b B \to T_{f(b)} M / V_b$ induced by $T_b f$ being surjective, which is equivalent to $V_b + \mathrm{im}\, T_b f = T_{f(b)} M$. This proves the equivalence.

If f is transverse to D, then D can be pulled back along f to a Dirac structure $\mathfrak{B}_f D$ on B (see e.g. [Bur13, Proposition 5.6]). Let ϕ be a spinor for D. By the first part, $f^*\phi$ is nowhere vanishing. Elements of $\mathfrak{B}_f D$ have the form $(v, (Tf)^*\eta)$, where $(Tf(v), \eta) \in D$. This implies that

$$(v, (Tf)^*\eta) \cdot f^*\phi = (Tf)^*((Tf(v), \eta) \cdot \phi) = 0.$$

So at every point $\mathfrak{B}_f L$ is included in the isotropic subspace associated to $f^*\phi$. Since the first vector space is maximally isotropic, the two must be equal. So $f^*\phi$ must be a spinor for $\mathfrak{B}_f D$ and f is a backward Dirac map. □

Lemma 4.4.1 immediately implies that Dirac structures always lift to the blowup of transversals.

Theorem 4.4.2 *Let D be a twisted Dirac structure on M. For any closed, embedded transversal $N \subseteq M$, D lifts to* $\mathrm{Blup}(M, N)$ *and the blowdown map is a backward Dirac map.*

Proof. Since the blowdown map p satisfies $T_{p(\xi)}N \subseteq \operatorname{im} T_\xi p$, for all $\xi \in \mathbb{P}(\nu_N(M))$, if follows that p is transverse to D. The claim follows from Lemma 4.4.1. □

4.5 The transverse and invariant dichotomy

In this section, we prove a surprising result.

Theorem 4.5.1 *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed, embedded, and connected submanifold of codimension > 1. If D lifts to a twisted Dirac structure on* $\mathrm{Blup}(M, N)$*, then one of the following two conditions holds.*

- *$N \subseteq M$ is a transversal. In this case, the blowdown map is a backward Dirac map.*
- *$N \subseteq M$ is an invariant submanifold. In this case, the blowdown map is a forward Dirac map.*

Motivated by Lemma 4.3.5, we treat the extension problem of line bundles in the next subsection first, and then return to the proof of Theorem 4.5.1.

4.5.1 Extending line bundles along codimension one submanifolds

Aiming to understand the question of extending spinor lines, in this subsection we provide necessary and sufficient conditions for when a rank one subbundle of a vector bundle, defined away from a codimension one submanifold, extends smoothly along the submanifold.

We first make precise the notion of vanishing order. Let $E \to M$ be a vector bundle, and denote by $J^kE \to M$ the **k-th order jet bundle** of E. This is again a vector bundle over M with fibre at $q \in M$ given by

$$J^k_qE = \Gamma(E)/\mathcal{I}_q^{k+1}\Gamma(E),$$

where $\mathcal{I}_q$ the ideal of functions vanishing at q. Then any smooth section $s \in \Gamma^\infty(E)$ induces a smooth section $j^k(s) \in \Gamma^\infty(J^kE)$, called the **$k$-th order jet of** s, via

$$j^k(s)(q) := s + \mathcal{I}_q^{k+1}\Gamma(E) \in J^k_qE.$$

Definition 4.5.2 The **vanishing order** of a section $s \in \Gamma(E)$ at $q \in M$ is the smallest $k \in \mathbb{N}_0$ such that $j^k(s)(q) \neq 0$. We say that s has **infinite vanishing** order at q if $j^k(s)(q) = 0$ for all k.

The following result will be used in Sections 4.5.2 and 4.6.

Lemma 4.5.3 *Let $E \to M$ be a vector bundle, $X \subseteq M$ a closed, embedded, and connected submanifold of codimension one. Let $s \in \Gamma(E)$ be given, satisfying $s^{-1}(0) \subseteq X$.*

1. *If s has constant finite vanishing order along X, then $\operatorname{span}(s)\big|_{M\setminus X}$ extends along X to a smooth subbundle of E of rank one.*
2. *If $\operatorname{span}(s)\big|_{M\setminus X}$ extends along X to a smooth subbundle of E of rank one, and there exists a nowhere vanishing section $\beta \in \Gamma(E^*)$ such that $\beta(s)$ has constant finite vanishing order along X, then s has (possibly lower) constant vanishing order along X.*

For the proof of Lemma 4.5.3 we need an auxiliary statement about quotients of smooth functions.

Lemma 4.5.4 *Let M be a manifold and $X \subseteq M$ a closed, embedded, and connected submanifold of codimension one. Suppose that $f \in \mathscr{C}^\infty(M)$ has constant vanishing order $k \in \mathbb{N}_0$ along X. If $g \in \mathscr{C}^\infty(M)$ divides f, then g has constant vanishing order $\ell \leq k$ along X.*

Proof. We can argue locally, and assume that $M = \mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ and that $X = \{x_1 = 0\}$. Since f vanishes to order k along X, by Hadamard's Lemma we can write

$$f = x_1^k \tilde{f}$$

for some $\tilde{f} \in \mathscr{C}^\infty(\mathbb{R}^n)$. Since $j^k f$ vanishes nowhere along X, it follows that $\tilde{f}\big|_X$ vanishes nowhere on X. Let U be a neighbourhood of X on which $\tilde{f}$ is invertible.

Let $g \in \mathscr{C}^\infty(\mathbb{R}^n)$ divide f. Since $\tilde{f}\big|_U$ is invertible, there is $h \in \mathscr{C}^\infty(U)$ such that

$$x_1^k = g \cdot h \quad \text{on} \quad U.$$

For any $y \in X$ and a function $a \in \mathscr{C}^\infty(U)$, let $o_y(a)$ denote the vanishing order of the function $x_1 \mapsto a(x_1, y)$ at $x_1 = 0$; in other words $o_y(a)$ is the degree of the first Taylor term with non-zero coefficient. Then o_y is multiplicative $o_y(ab) = o_y(a) + o_y(b)$. Therefore, we obtain that:

$$\forall\, y \in X \;:\; o_y(g) + o_y(h) = k.$$

On the other hand, note that for any $a \in \mathscr{C}^\infty(U)$ the function $y \mapsto o_y(a)$ is locally non-increasing.

Since X is connected, this implies that $o_y(g)$ and $o_y(h)$ are constant. So there is $0 \leq \ell \leq k$ such that $o_y(g) = \ell$, for all $y \in X$. Thus, we can write $g = x_1^\ell \tilde{g}$, where $\tilde{g}\big|_X$ is a nowhere vanishing function. Hence, g has constant vanishing order ℓ along X. □

Proof of Lemma 4.5.3. First note that if an extension of $\operatorname{span}(s)$ to X exists, it is necessarily unique since $M \setminus X$ is dense. For the first part, assume that s

has constant vanishing order k along X. Then for any local defining function x of X, we have that

$$\tilde{s} = \tfrac{s}{x^k}.$$

extends along X to a smooth section of E, and does not vanish anywhere in a neighbourhood of X. Hence, $\tilde{s}$ generates a line bundle that gives the desired extension.

For the second part, suppose $\tilde{s} \in \Gamma(E)$ is a (local) frame of the extension of $\operatorname{span}(s)\big|_{M\setminus X}$. Then there exists a function $g \in \mathscr{C}^\infty(M)$ with

$$s = g\tilde{s}.$$

Thus, $\beta(s) = g\beta(\tilde{s})$, and so g divides $\beta(s)$. If $\beta(s)$ has constant vanishing order k along X, by Lemma 4.5.4 g has constant vanishing order ℓ along X for some $\ell \leq k$. Then, from $s = g\tilde{s}$ and since $\tilde{s}$ does not vanish on X by assumption, the statement follows. ☐

The following consequence of our techniques will be used to show that Dirac structures can be extended to an open and dense subset inside the blowup.

Lemma 4.5.5 *Let $E \to M$ be a vector bundle and $X \subseteq M$ a closed embedded submanifold of codimension one. Let $s \in \Gamma(E)$ be a section such that $s^{-1}(0) \subseteq X$ and $j(s)$ is nowhere vanishing. There exists an open dense set $V \subseteq X$ such that* $\operatorname{span}(s)\big|_{M\setminus X}$ *extends to a line bundle over $(M \setminus X) \cup V$.*

Proof. From the proof of Lemma 4.5.3 we see that, if s has constant vanishing order $\ell \geq 0$ along an open subset $V_\ell \subseteq X$, then $\operatorname{span}(s)\big|_{M\setminus X}$ extends to $(M\setminus X) \cup V_\ell$. In particular, we can choose

$$V_\ell := C_\ell^\circ \setminus C_{\ell+1}, \quad \text{where} \quad C_0 := X, \quad C_{\ell+1} := j^\ell s\big|_X^{-1}(0),$$

and $\cdot^\circ$ the interior as a subset of X. Hence, if we set $V = \bigcup_{\ell=0} V_\ell$, then $\operatorname{span}(s)\big|_{M\setminus X}$ extends to a smooth line bundle on $(M \setminus X) \cup V$.

It is left to show that V is dense in X. First, using that $C_{\ell+1} \subseteq C_\ell$, that $C_0 = X$, and that, by assumption, $\bigcap_{\ell=0}^\infty C_\ell = \emptyset$, we obtain the following decomposition of X into disjoint subsets:

$$X = \bigcup_{\ell\geq 0} C_\ell \setminus C_{\ell+1}.$$

Using that $V_\ell \subseteq C_\ell$, we have that

$$X\setminus V = \bigcup_{\ell\geq 0} (C_\ell\setminus C_{\ell+1})\setminus V_\ell = \bigcup_{\ell\geq 0} C_\ell\setminus(C_\ell^\circ \cup C_{\ell+1}) = \bigcup_{\ell\geq 0} \partial C_\ell\setminus C_{\ell+1}.$$

Since all C_ℓ are closed, their boundaries are closed sets with no inner points. By Baire's Theorem, their union still has no inner points. The same holds also for $\cup_{\ell\geq 0}\partial C_\ell\setminus C_{\ell+1}$, and so, V is dense. ☐

Lemma 4.5.5 implies the following result.

Corollary 4.5.6 *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed and embedded submanifold. There exists an open and dense subset $V \subseteq \mathbb{P}(\nu_N(M))$ such that the Dirac structure $p^*(D|_{M\setminus N})$ extends smoothly to a Dirac structure on $V \cup \big(\mathrm{Blup}(M,N) \setminus \mathbb{P}(\nu_N(M))\big)$.*

Proof. Let Σ be the spinor line on $\big(\mathrm{Blup}(M,N) \setminus \mathbb{P}(\nu_N(M))\big)$ of $p^*(D|_{M\setminus N})$. Using the standard charts on the blowup, in which the blowdown map is algebraic (4.2), it is easy to check that for any (local) spinor ϕ for D, we have that $j^\infty p^*(\phi)$ does not vanish anywhere. Then Lemma 4.5.5 provides an open and dense subset $V \subseteq \mathbb{P}(\nu_N(M))$ over which Σ extends smoothly. Lemma 4.3.5 implies that $p^*(D|_{M\setminus N})$ extends to a p^*H-twisted Dirac structure to $V \cup \big(\mathrm{Blup}(M,N) \setminus \mathbb{P}(\nu_N(M))\big)$. □

4.5.2 Proof of Theorem 4.5.1

We first prove a pointwise version of Theorem 4.5.1.

Lemma 4.5.7 *Let D be a Dirac structure on M and $N \subseteq M$ a closed, embedded submanifold of codimension > 1. If D lifts to* Blup(M,N)*, then at any $q \in N$ either of the following conditions holds.*

- $\mathrm{pr}_{TM}(D_q) \subseteq T_qN$;
- $\mathrm{pr}_{TM}(D_q) + T_qN = T_qM$.

Proof. As explained in Subsection 4.3.4, it suffices to work locally, with an untwisted Dirac structure.

Assume that D lifts, but neither condition holds at $q \in N$. The failure of the first condition implies the existence of some $v + \alpha \in \Gamma(D)$ such that $v_q \notin T_qN$.

We follow the first steps in [Blo17] of the proof of Theorem 4.3.7. By [Blo17, Lemma 3.4], there exists a closed 2-form ω defined around $q \in M$ such that $\mathrm{i}_v\omega = \alpha$. Consequently, around q the Dirac structure D is isomorphic to $\exp^{-\omega}D$, which contains $v + 0$. Hence, we can assume $v \in \Gamma(D)$ from the start. By using a chart around q adapted to N, we find a small neighbourhood U of q in M and a codimension-one submanifold $\iota\colon \tilde{M} \hookrightarrow M$ through q, such that $N \cap U \subseteq \tilde{M}$ and such that $v_q \notin T_q\tilde{M}$. By [Blo17, Lemma 3.5], after shrinking U and $\tilde{M}$, we find a diffeomorphism $U \cong (-\varepsilon, \varepsilon) \times \tilde{M}$, which sends $\tilde{M}$ to $\{0\} \times \tilde{M}$, $v|_U$ to $\frac{\partial}{\partial t}$, and the Dirac structure $D|_U$ to the product Dirac structure

$$T(-\varepsilon,\varepsilon) \times \tilde{D},$$

where $\tilde{D} = \mathfrak{B}_\iota D$. We replace M by this neighbourhood. Then, because of the product structure, the spinor ϕ_D of D may be taken to be the spinor $\phi_{\tilde{D}}$ of $\tilde{D}$. In particular, ϕ_D is independent of t and $\mathrm{d}t$.

Since the second condition does not hold, and $(\mathrm{pr}_{TM}(D_q))^{\mathrm{ann}} = D \cap T_q^*M$, there exists $\xi \neq 0$, such that $\xi \in D_q \cap (T_qN)^{\mathrm{ann}}$. Since $\xi, \frac{\partial}{\partial t}\big|_q \in D$, we have that $\xi(\frac{\partial}{\partial t}) = 0$, and therefore $\xi|_{T_q\tilde{M}} \neq 0$. So there exists a function

$x_2 \in \mathscr{C}^\infty(\tilde{M})$ with $x_2\big|_N = 0$ and $\xi = \mathrm{d}x_2\big|_q$. By completing $\{x_2\}$ to a chart on $\tilde{M}$ which is adapted to N and centred at q, we obtain coordinates $(t, x_2, \dots, x_k, y_{k+1}, \dots, y_m)$ in which $N = \{t = x_2 = \cdots = x_k = 0\}$ and $t(q) = x_i(q) = y_j(q) = 0$.

In the constructed chart, the spinor ϕ_D of D decomposes uniquely as

$$\phi_D = \phi_0 + \phi_1 \wedge \mathrm{d}x_2,$$

where ϕ_0 has no $\mathrm{d}x_2$-contribution. We still have that ϕ_0, ϕ_1 are independent of t and $\mathrm{d}t$, and, since $\mathrm{d}x_2\big|_q \in D$, by the definition of the action ρ in Section 4.3.2, we have $\phi_0(0) = 0$. Next, we use multi-indices to write

$$\phi_0 = \sum_{I,J} \phi_0^{I,J} \mathrm{d}x_I \wedge \mathrm{d}y_J \quad \text{and} \quad \phi_1 = \sum_{I,J} \phi_1^{I,J} \mathrm{d}x_I \wedge \mathrm{d}y_J.$$

We pull the spinor ϕ_D back to the chart U_t on $\mathrm{Blup}(M, N)$, i.e. along the map $p(\tilde{t}, \tilde{x}, y) = (\tilde{t}, \tilde{t}\tilde{x}, y)$, where $\tilde{x} = (\tilde{x}_2, \dots, \tilde{x}_k)$ and $y = (y_{k+1}, \dots, y_m)$ (see (4.2)). Then we obtain

$$\begin{aligned} p^*\phi_D\big|_{U_t} = &\sum_{I,J} \Big(\tilde{t}^{|I|} p^* \phi_0^{I,J} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J + \tilde{t}^{|I|+1} p^* \phi_1^{I,J} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \wedge \mathrm{d}\tilde{x}_2 \\ &+ \mathrm{d}\tilde{t} \wedge \Big(\sum_{u=3}^{k} \tilde{t}^{|I|-1} p^* \phi_0^{I,J} \tilde{x}_u \mathrm{i}_{\partial_{\tilde{x}_u}} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \Big) \\ &+ \mathrm{d}\tilde{t} \wedge \Big(\sum_{u=3}^{k} \tilde{t}^{|I|} p^* \phi_1^{I,J} \tilde{x}_u \mathrm{i}_{\partial_{\tilde{x}_u}} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \wedge \mathrm{d}\tilde{x}_2 \\ &+ (-1)^{|I|+|J|} \tilde{x}_2 \tilde{t}^{|I|} p^* \phi_1^{I,J} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \Big) \Big). \end{aligned}$$

Recall that $U_t \cap \mathbb{P}(\nu_N(M))) = \{\tilde{t} = 0\}$. Since by assumption the Dirac structure lifts, there exists a function $f \in \mathscr{C}^\infty(U_t \setminus \{\tilde{t} = 0\})$ such that $fp^*\phi_D\big|_{U_t}$ extends smoothly along $\{\tilde{t} = 0\}$, and vanishes nowhere on U_t. In particular, for $(\tilde{x}, y) = 0$ and $\tilde{t} \neq 0$, using that $\phi_0(0) = 0$, we see that all terms in of $p^*\phi_D\big|_{U_t}$ in the sum above vanish identically along the line $(\tilde{x}, y) = 0$, except for

$$\sum_{I,J} \tilde{t}^{|I|+1} p^* \phi_1^{I,J}(0) \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \wedge \mathrm{d}\tilde{x}_2.$$

Hence, there must exist index sets I_*, J_* such that the smooth function

$$g := f\tilde{t}^{|I_*|+1} p^* \phi_1^{I_*,J_*} \in \mathscr{C}^\infty(U_t)$$

satisfies $g(0) \neq 0$. By shrinking U_t, we may assume that g doesn't vanish anywhere on U_t, and by replacing f by f/g, we may assume that $f\tilde{t}^{|I_*|+1} p^* \phi_1^{I_*,J_*} = 1$ on U_t. Hence,

$$f = \frac{1}{\tilde{t}^{|I_*|+1} p^* \phi_1^{I_*,J_*}}.$$

However, this form of f implies that $fp^*\phi_D\big|_{U_t}$ has a singularity at $\tilde{t} = 0$. Indeed, for $\tilde{t} \neq 0$, along the submanifold $\{x_3 = \ldots = x_k = 0\}$, we have

$$\begin{aligned} fp^*\phi_D =& \frac{1}{\tilde{t}^{|I_*|+1} p^*\phi_1^{I_*,J_*}} \sum_{I,J} \Big(\tilde{t}^{|I|} p^*\phi_0^{I,J} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \\ &+ \tilde{t}^{|I|+1} p^*\phi_1^{I,J} \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \wedge \mathrm{d}\tilde{x}_2 \\ &+ (-1)^{|I|+|J|} \tilde{x}_2 \tilde{t}^{|I|} p^*\phi_1^{I,J} \mathrm{d}\tilde{t} \wedge \mathrm{d}\tilde{x}_I \wedge \mathrm{d}y_J \Big). \end{aligned}$$

The coefficient of the term $\mathrm{d}\tilde{t} \wedge \mathrm{d}\tilde{x}_{I_*} \wedge \mathrm{d}y_{J_*}$ is given by

$$\frac{\tilde{x}_2}{\tilde{t}} \mathrm{d}\tilde{t} \wedge \mathrm{d}\tilde{x}_{I_*} \wedge \mathrm{d}y_{J_*},$$

which is not well-defined in $\tilde{t} = 0$. Hence, we obtain a contradiction to the assumption that the Dirac structure lifts. □

Next, we use Lemma 4.5.3 to show that, if D lifts over a point where the Dirac structure is tangent to the submanifold, then locally the pulled back spinor has constant vanishing order.

Lemma 4.5.8 *Let D be a Dirac structure on M, $N \subseteq M$ a closed and embedded submanifold, and $q \in N$ such that*

$$\mathrm{pr}_{TM}(D_q) \subseteq T_q N. \tag{4.5}$$

Let ϕ denote a spinor corresponding to D defined around q. The following are equivalent.

- *The Dirac structure $p^*(D\big|_{M\setminus N})$ extends to a neighbourhood of $p^{-1}(q) \subseteq \mathbb{P}(\nu_N(M))$.*
- *The order of vanishing of $p^*\phi$ along $\mathbb{P}(\nu_N(M))$ is constant around $p^{-1}(q)$.*

Proof. If $p^*\phi$ has constant vanishing order along $\mathbb{P}(\nu_N(M))$, then the first part of Lemma 4.5.3 together with Lemma 4.3.5 imply that D lifts in a neighbourhood of $p^{-1}(q)$.

For the converse, we apply the second part of Lemma 4.5.3. In order to do so, we first find coordinates that are adapted to both D and N.

By Theorem 4.3.7 (see also Subsection 4.3.4) we can assume that $M = U \times Z$, $q = (q_U, q_Z)$, and $D = \mathrm{graph}(\pi) \times TZ$ for a Poisson structure π on U vanishing at q_U.

The assumption (4.5) implies that $\mathrm{pr}_Z\big|_N \colon N \to Z$ is a submersion at q. After shrinking U, one can split $U = V \times W$, with $q_U = (q_V, q_W)$, such that $T_{q_W}W = \ker T_q(\mathrm{pr}_Z\big|_N)$. Then the projection $\mathrm{pr}\colon V \times W \times Z \to W \times Z$ induces a linear isomorphism

$$T_q\mathrm{pr}\big|_N \colon T_qN \xrightarrow{\sim} T_{q_W}W \oplus T_{q_Z}Z.$$

By the inverse function theorem, $\mathrm{pr}\big|_N \colon N \to W \times Z$ is a diffeomorphism in an open neighbourhood of q. Therefore, after shrinking all neighbourhoods, we find a smooth map $f \colon W \times Z \to V$, such that

$$(\mathrm{pr}\big|_N)^{-1}(w,z) = (f(w,z), w, z).$$

Denote by $v = (v_1, \ldots, v_k)$, $w = (w_1, \ldots, w_l)$, and $z = (z_1, \ldots, z_m)$ coordinates on V, W, and Z, around q_V, q_W, and q_Z, respectively. Define $x_i := v_i - f_i(w,z)$, $i = 1, \ldots, k$, and $x = (x_1, \ldots, x_k)$. Then (x, w, z) is a chart around q on M which is a submanifold chart for N, i.e. $N = \{x = 0\}$.

In the first set of coordinates, a spinor of D is given by

$$\begin{aligned}\phi =& \exp^{\mathrm{i}_\pi}(\mathrm{d}v_1 \wedge \ldots \wedge \mathrm{d}v_k \wedge \mathrm{d}w_1 \wedge \ldots \wedge \mathrm{d}w_l) \\ =& \mathrm{d}v_1 \wedge \ldots \wedge \mathrm{d}v_k \wedge \mathrm{d}w_1 \wedge \ldots \wedge \mathrm{d}w_l + \text{ lower degree forms}\end{aligned}.$$

Writing ϕ in the second set of coordinates, we obtain

$$\phi = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k \wedge \mathrm{d}w_1 \wedge \ldots \wedge \mathrm{d}w_l + \text{ other terms},$$

where the “other terms” are either or lower form degree or contain at least one $\mathrm{d}z_j$.

For $i = 1, \ldots, k$ consider the chart U_i on $\mathbb{P}(\nu_N(M))$ with coordinates $(\tilde{x}, w, z)$, defined in (4.2). Then we have that

$$p^*\phi\big|_{U_{x_i}} = \tilde{x}_i^{k-1}\mathrm{d}\tilde{x}_1 \wedge \ldots \wedge \mathrm{d}\tilde{x}_k \wedge \mathrm{d}w_1 \wedge \ldots \wedge \mathrm{d}w_l + \text{ other terms}, \tag{4.6}$$

where again the “other terms” are either of lower degree of contain at least one $\mathrm{d}z_j$. Hence, the multivector field

$$\beta = \frac{\partial}{\partial \tilde{x}_1} \wedge \ldots \wedge \frac{\partial}{\partial \tilde{x}_k} \wedge \frac{\partial}{\partial w_1} \wedge \ldots \wedge \frac{\partial}{\partial w_l},$$

satisfies that $\beta(p^*\phi)$ has constant vanishing order equal to $k-1$ along $\{\tilde{x}_i = 0\} = \mathbb{P}(\nu_N(M)) \cap U_i$. So if the line bundle spanned by $p^*\phi$ extends to a neighbourhood of $p^{-1}(q)$, then the second part of Lemma 4.5.3 implies that $p^*\phi$ has constant vanishing order in a neighbourhood of $p^{-1}(q)$. □

The following is used to show that the blowdown map is a forward Dirac map.

Lemma 4.5.9 *Let D be a Dirac structure on M and $N \subseteq M$ a closed, embedded submanifold of codimension > 1. Assume that D lifts to the Dirac structure $\tilde{D}$ on* $\mathrm{Blup}(M,N)$*. Then the blowdown map is a forward Dirac map if and only if N is an invariant submanifold.*

Proof. Suppose $p \colon \mathrm{Blup}(M,N) \to M$ is a forward Dirac map. For any $q \in N$ we have

$$\mathrm{pr}_{TM}(D_q) \subseteq \bigcap_{\xi \in p^{-1}(q)} \mathrm{im}\, T_\xi p = T_q N.$$

Hence, N is invariant.

Conversely, assume that N is invariant. We have to show that for all $\xi \in \mathbb{P}(\nu_N(M))$, $\mathfrak{F}_p(\tilde{D}_\xi) = D_{p(\xi)}$. Fix $(z, \eta) \in D_{p(\xi)}$. Let $(X, \alpha) \in \Gamma(D)$ with $(X, \alpha)\big|_{p(\xi)} = (z, \eta)$. Since X is tangent to N, by Lemma 4.2.1, it has a unique lift $\tilde{X}$ to $\mathrm{Blup}(M, N)$. Then $(\tilde{X}, p^*\alpha) \in \Gamma(\mathbb{T}\mathrm{Blup}(M, N))$ maps the dense subset $\mathrm{Blup}(M, N) \setminus \mathbb{P}(\nu_N(M))$ into $\tilde{D}$. Thus, $(\tilde{X}, p^*\alpha) \in \Gamma(\tilde{D})$, implying that

$$\begin{pmatrix} (T_\xi p)(\tilde{X}(\xi)) \\ \alpha(p(\xi)) \end{pmatrix} \in \mathfrak{F}_p(\tilde{D}_\xi).$$

As $\tilde{X}$ p-projects to X, we see that the above pair is just $(X, \alpha)\big|_{p(\xi)} = (z, \eta)$. This shows $\mathfrak{F}_p(\tilde{D}_\xi) \supseteq D_{p(\xi)}$. Equality follows because both are maximally isotropic subspaces of $\mathbb{T}_{p(\xi)}M$. □

Summarising, we can prove Theorem 4.5.1.

Proof of Theorem 4.5.1. By Lemma 4.5.7, at any $q \in N$, D is either tangent or transverse to N. These two cases are mutually exclusive as $\operatorname{codim} N > 0$. Moreover, the set where N is transverse to D is open in N, because the condition $T_qN + \mathrm{pr}_{TM}(D_q) = T_qM$ is open. Since N is connected, it suffice to show that the tangential points also form an open subset. Indeed, let $q \in N$ be such that $\mathrm{pr}_{TM}(D_q) \subseteq T_qN$ and let ϕ be a spinor for D defined around q. By Lemma 4.4.1, $p^*\phi$ must vanish along $p^{-1}(q)$ as q is not a transverse point. But then, by Lemma 4.5.8, $p^*\phi$ must vanish also in a neighbourhood U of $p^{-1}(q) \subseteq \mathbb{P}(\nu_N(M))$. Then $p(U) \subseteq N$ is a open neighbourhood of q in N, and D and N cannot be transverse at any point $q' \in p(U)$, again because of Lemma 4.4.1. Therefore, Lemma 4.5.7 shows that $p(U)$ consists entirely of points where D is tangent to N.

Finally, if N is a transversal, Lemma 4.4.1 shows that p is a backward Dirac map, and if N is invariant, then Lemma 4.5.9 shows that p is a forward Dirac map. □

4.6 Blowup of invariant submanifolds

After the previous two sections, it remains to characterise the invariant submanifolds $N \subseteq M$ of a twisted Dirac structure D for which D lifts to the blowup $\mathrm{Blup}(M, N)$. Recall from the Introduction that an invariant submanifold $N \subseteq M$ of D yields a short exact sequence of Lie algebroids:

$$0 \longrightarrow (TN)^{\mathrm{ann}} \longrightarrow D\big|_N \longrightarrow \mathfrak{B}_{\iota_N} D \longrightarrow 0.$$

In particular, $(TN)^{\mathrm{ann}}$ is a bundle of Lie algebras over N.

Recall also from the Introduction that, given a Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$, the **height** of an element $\xi \in \mathfrak{g}^* \setminus \{0\}$ is defined as the integer $k \in \mathbb{N}_0$ such that

$$\xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^k \neq 0 \quad \text{and} \quad \xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} = 0. \tag{4.7}$$

The Lie algebra $\mathfrak{g}$ is called **a Lie algebra of constant height** k if (4.7) holds for all $\xi \in \mathfrak{g}^* \setminus \{0\}$.

We can state now the main result of this section.

Theorem 4.6.1 *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed, embedded and connected submanifold, which is invariant. The following are equivalent.*

- *D lifts to* $\operatorname{Blup}(M,N)$.
- *The Lie algebras $(T_qN)^{\mathrm{ann}}$, $q \in N$, have all the same constant height k.*

Example 4.6.2 On a Lie group G with a bi-invariant (not necessarily positive) metric $(\cdot\,,\cdot)$, there is a canonical twisted Dirac structure D, called Cartan-Dirac structure [ŠW01, Example 5.2]. The twist is provided by the corresponding Cartan 3-form, i.e. the bi-invariant 3-form on G which at the unit e reads $H(u,v,w) := -\frac{1}{2}([u,v],w)$ for $u,v,w \in \mathfrak{g} = T_eG$. In a neighbourhood U of the unit e, the Cartan-Dirac structure D is the graph of an $H\big|_U$-twisted Poisson structure π. At every $g \in U$, the map $\pi_g^\sharp \colon T_g^*G \to T_gG$ is obtained by left-translating $2(\operatorname{Ad}_g - 1)(\operatorname{Ad}_g + 1)^{-1} \colon \mathfrak{g} \to \mathfrak{g}$ and using the identification $\mathfrak{g}^* \cong \mathfrak{g}$ induced by the metric. It is known that the induced Lie algebroid structure on D is isomorphic over id_G to the transformation Lie algebroid associated to the action of G on itself by conjugation. In particular, the unit element $\{e\}$ of G constitutes a leaf of D, and the isotropy Lie algebra of D at e is just $\mathfrak{g}$.

Now choose $G = SO(3)$, endowed with any bi-invariant Riemannian metric (which exists because the Lie group is compact). Theorem 4.8.1 below shows that $\mathfrak{so}(3)$ is of constant height 1. Hence, by Theorem 4.6.1, D lifts to an p^*H-twisted Dirac structure on $\operatorname{Blup}(G,\{e\})$, which is not given by the graph of a bivector field around $p^{-1}(e)$.

Theorem 4.6.1 is a consequence of the results of the following subsections, in which we give alternative criteria for the liftability property.

4.6.1 The constant vanishing order criterion

The following criterion for a Dirac structure to lift to the blowup is an immediate consequence of Lemma 4.5.8 and its proof.

Corollary 4.6.3 *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed, embedded and connected submanifold, which is invariant and has codimension > 1. The following are equivalent.*

- *D lifts to a p^*H-twisted Dirac structure on* $\operatorname{Blup}(M,N)$.
- *There exists $\ell \in \{1,\dots,\operatorname{codim} N - 1\}$ such that, for any (local) spinor ϕ for D, $p^*\phi$ has constant vanishing order ℓ along $\mathbb{P}(\nu_N(M))$.*

In the Poisson category, we have the following version of this result.

Corollary 4.6.4 *Let (M,π) be Poisson and $N \subseteq M$ a closed and embedded Poisson submanifold. The following are equivalent.*

- *π lifts to a Poisson structure $\tilde{\pi}$ on* $\operatorname{Blup}(M,N)$.

- *The spinor* $p^*(\exp^\pi \lambda)$ *has constant vanishing order* $\ell = \operatorname{codim} N - 1$ *along* $\mathbb{P}(\nu_N(M))$*, for any (local) volume form* λ *on* M*.*

Proof. If $\operatorname{codim}(N) = 1$, there is nothing to show. Let $\operatorname{codim}(N) > 1$. By Corollary 4.6.3, graph(π) lifts to a Dirac structure $\tilde{D}$, exactly when the spinor $\phi^*(\exp^\pi \lambda)$ has constant vanishing order ℓ. In this case, $t^{-\ell}\phi^*(\exp^\pi \lambda)$ is a nowhere vanishing spinor corresponding to $\tilde{D}$, where t is any local coordinate defining $\mathbb{P}(\nu_N(M))$, i.e. locally $\mathbb{P}(\nu_N(M)) = \{t = 0\}$. On the other hand, $t^{-\ell}\phi^*(\exp^\pi \lambda)$ corresponds to a Poisson structure, exactly when its component of top degree, i.e. $t^{-\ell}\phi^*(\lambda)$, is nowhere vanishing. Using the standard charts on the blowup, as in (4.6), we see that $\phi^*(\lambda)$ has constant vanishing order $\operatorname{codim} N - 1$. This implies the equivalence. □

Example 4.6.5 Consider the vector bundle $E = \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$. Denote its canonical global frame by $\{e_1, e_2, e_3\}$ and denote by (y_1, y_2) the variables of the base. Let $f \in \mathscr{C}^\infty(\mathbb{R}^2)$ be a function. Equip the fibre $E_{(y_1,y_2)}$ with the Lie bracket

$$[e_i, e_j]_{(y_1,y_2)} = f(y_1, y_2) \sum_{k=1}^{3} \varepsilon_{ijk} e_k,$$

where $\varepsilon_{ijk} = 1$ if ijk is a cyclic permutation of 123, $\varepsilon_{ijk} = -1$ for a cyclic permutation of 213, and $\varepsilon_{ijk} = 0$ otherwise. Then $E \to \mathbb{R}^2$ is a bundle of Lie algebras, with fibre

$$E_{(y_1,y_2)} \simeq \begin{cases} \mathfrak{so}(3) & \text{if } f(y_1, y_2) \neq 0 \\ \text{abelian } \mathbb{R}^3 & \text{if } f(y_1, y_2) = 0. \end{cases}$$

On E^*, we obtain a Poisson structure π with global spinor line generated by

$$\phi_\pi = f(y_1, y_2) \sum_{i=1}^{3} x_i \mathrm{d}x_i \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 + \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2,$$

where we denote the linear fibre coordinates induced by (the dual frame to) $\{e_i\}_i$ by (x_1, x_2, x_3).

1. Let N be the zero section of the vector bundle E^*. The Dirac structure graph(π) lifts to the blowup Blup(E^*, N) exactly when $f = 0$ or f vanishes nowhere. Indeed, for $i \in \{1, 2, 3\}$,

$$\begin{aligned}(p^*\phi_\pi)\big|_{U_i} =& f(y_1, y_2)\Big(\sum_{j\neq i} \tilde{x}_i^2 \tilde{x}_j \mathrm{d}\tilde{x}_j + \tilde{x}_i\Big(1 + \sum_{j\neq i} \tilde{x}_j^2\Big)\mathrm{d}\tilde{x}_i\Big) \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 \\ &+ \tilde{x}_i^2 \mathrm{d}\tilde{x}_1 \wedge \mathrm{d}\tilde{x}_2 \wedge \mathrm{d}\tilde{x}_3 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2.\end{aligned}$$

Let $q = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, y_1, y_2) \in \mathbb{P}(E^*) \cap U_i$, so $\tilde{q}_i = 0$, with vanishing ideal $\mathcal{I}_q$. Reducing the coefficients in $(p^*\phi_\pi)\big|_{U_i}$ modulo $\mathcal{I}_q^2$, the only term that might be non-zero is

$$f(y_1, y_2)\, \tilde{x}_i\, (1 + \sum_{j\neq i} \tilde{q}_j^2) \in \mathcal{I}_q.$$

This belongs to $\mathcal{I}_q^2$ precisely when $f(y_1, y_2) = 0$. Hence, the vanishing order of $(p^*\phi_\pi)\big|_{U_i}$ is constant along $\mathbb{P}(E^*) \cap U_i = \{\tilde{x}_i = 0\}$ if and only if $f = 0$ or f vanishes nowhere.

If $f = 0$, the vanishing order is $2 = \operatorname{codim} N - 1$, $\pi = 0$, and the lifted Dirac structure corresponds to the zero Poisson structure.

If f vanishes nowhere, $(p^*\phi_\pi)\big|_{U_i}$ has vanishing order 1 along $\mathbb{P}(\nu_N(M)) \cap U_i$ and so graph(π) lifts to a Dirac structure, which does not come from a Poisson structure.

2. Suppose that f vanishes at the origin $0 \in \mathbb{R}^2$, and take N to be the fibre $E^*\big|_0$. Then, by Corollary 4.6.4, π lifts to a Poisson structure on $\operatorname{Blup}(E^*, N)$. Indeed, for $j \in \{1, 2\}$,

$$(p^*\phi_\pi)\big|_{U_j} = \tilde{y}_j \Big(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \sum_{i=1}^{3} (p^* f)\big|_{U_j} x_i \mathrm{d}x_i\Big) \wedge \mathrm{d}\tilde{y}_1 \wedge \mathrm{d}\tilde{y}_2$$

has constant vanishing order $\operatorname{codim} N - 1 = 1$ along $\mathbb{P}(\nu_N(E^*)) \cap U_j$.

4.6.2 Reducing to bundles of linear Poisson structures

As a first step to prove Theorem 4.6.1, we reduce the problem to the setting of linear Poisson structures.

Given a bundle of Lie algebras $(\underline{\mathfrak{g}}, [\,\cdot\,,\,\cdot\,])$ over N, the dual vector bundle $r\colon \underline{\mathfrak{g}}^* \to N$ comes equipped with a with a fibrewise linear Poisson structure π_{lin}, with Poisson bracket determined by:

$$\{\mathrm{ev}_\alpha, \mathrm{ev}_\beta\} = \mathrm{ev}_{[\alpha,\beta]}, \quad \{\mathrm{ev}_\alpha, r^* f\} = 0, \quad \{r^* f, r^* g\} = 0,$$

for all $\alpha, \beta \in \Gamma^\infty(\underline{\mathfrak{g}})$ and all $f, g \in \mathscr{C}^\infty(N)$, where $\mathrm{ev}_\alpha \in \mathscr{C}^\infty(\underline{\mathfrak{g}}^*)$ is α, viewed as a fibrewise linear map.

In particular, for an invariant submanifold $N \subseteq M$ of a Dirac structure D, the normal bundle $\nu_N(M) = ((TN)^{\mathrm{ann}})^*$ has a linear (untwisted) Poisson structure, denoted π_{lin}, coming from the bundle of Lie algebras on $(TN)^{\mathrm{ann}}$. Linearity implies that π_{lin} vanishes along $0_N \subseteq \nu_N(M)$, and so 0_N is an invariant submanifold for π_{lin}.

Theorem 4.6.6 *Let D be a twisted Dirac structure on M and $N \subseteq M$ a closed and embedded submanifold, which is invariant. The following are equivalent.*

- *D lifts to* $\operatorname{Blup}(M, N)$.
- $\operatorname{graph}(\pi_{\mathrm{lin}})$ *lifts to* $\operatorname{Blup}(\nu_N(M), 0_N)$.

In that case, the respective pullback spinors have the same vanishing orders along $\mathbb{P}(\nu_N(M))$.

Proof. As remarked in Subsection 4.3.4, it suffices to check locally whether D (respectively $\operatorname{graph}(\pi_{\mathrm{lin}})$) lifts to the blowup. In Subsection 4.3.4, we showed that isomorphisms of Dirac structures induce isomorphisms between the corresponding bundle of Lie algebras $(TN)^{\mathrm{ann}}$, and so, isomorphisms between the corresponding linear Poisson structures π_{lin}. Therefore, it suffices to work in the local product neighbourhoods from Corollary 4.3.8, as these have isomorphic linear Poisson structures.

So we assume that $M = X \times Y \times Z$, $D = \mathrm{graph}(\pi) \times TZ$, where π is a Poisson structure on $X \times Y$, and that $N = \{q_X\} \times Y \times Z$, where $\{q_X\} \times Y \subseteq X \times Y$ is a Poisson submanifold. Moreover, we may assume that $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$, $Z \subseteq \mathbb{R}^p$ are open subsets, with coordinates x, y and z, respectively, and that $q_X = 0$. Then we can write

$$\pi = \frac{1}{2}\sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{i,\alpha} \pi_{i\alpha} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_\alpha} + \frac{1}{2}\sum_{\alpha,\beta} \pi_{\alpha\beta} \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}, \tag{4.8}$$

where the coefficients are smooth functions of $(x, y) \in X \times Y$. Invariance of $\{0\} \times Y$ means that $\pi_{ij}(0, y) = 0$ and $\pi_{i\alpha}(0, y) = 0$. We can identify $\nu_N(M) = \mathbb{R}^m \times Y \times Z$ and then the Poisson structure π_{lin} becomes the linearisation of the first term:

$$\pi_{\mathrm{lin}} = \frac{1}{2}\sum_{i,j,k} x_k \frac{\partial \pi_{ij}}{\partial x_k}(0, y) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

We claim that $w := \pi - \pi_{\mathrm{lin}}$ can be lifted to a smooth bivector field $\tilde{w}$ on $\mathrm{Blup}(M, N)$. This follows from Lemma 4.2.1 because w can be written as a sum $w = \sum_k U_k \wedge V_k$, where U_k and V_k are vector fields tangent to $N = \{0\} \times Y \times Z$. This is clear for the last term in (4.8), for the middle it follows because $\pi_{i\alpha}(0, y) = 0$, and for the remainder it follows because its coefficients vanish quadratically along N.

Consider the differential forms

$$\lambda_X := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m, \quad \lambda_Y := \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n, \quad \text{and} \quad \lambda_Z := \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_p.$$

Spinors for $D = \mathrm{graph}(\pi) \times TZ$ and $\mathrm{graph}(\pi_{\mathrm{lin}})$ are given, respectively, by

$$\phi_D = \exp^{\pi}(\lambda_X \wedge \lambda_Y) = \exp^{w}((\exp^{\pi_{\mathrm{lin}}} \lambda_X) \wedge \lambda_Y) \quad \text{and}$$

$$\phi_{\pi_{\mathrm{lin}}} = (\exp^{\pi_{\mathrm{lin}}} \lambda_X) \wedge \lambda_Y \wedge \lambda_Z.$$

Their pullbacks to $\mathbb{B} := \mathrm{Blup}(X \times Y \times Z, \{0\} \times Y \times Z) \cong \mathrm{Blup}(X, \{0\}) \times Y \times Z$ are given by

$$\begin{aligned} p^*(\phi_D) =& p^*\big(\exp^{w}((\exp^{\pi_{\mathrm{lin}}} \lambda_X) \wedge \lambda_Y\big) \\ =& \exp^{\tilde{w}}\big(p^*\big(\exp^{\pi_{\mathrm{lin}}} \lambda_X) \wedge \lambda_Y\big) \quad \text{and} \\ p^*(\phi_{\pi_{\mathrm{lin}}}) =& p^*(\exp^{\pi_{\mathrm{lin}}} \lambda_X) \wedge \lambda_Y \wedge \lambda_Z. \end{aligned}$$

By Corollary 4.6.3, the first condition in the theorem is equivalent to the vanishing order of $p^*(\phi_D)$ being constant along $\mathbb{P}(\mathbb{R}^m) \times Y \times Z$. Since $\exp^{\tilde{w}}$ is an automorphism of the vector bundle $\wedge^\bullet T^*\mathbb{B}$, this condition is equivalent to $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda_X) \wedge \lambda_Y$ having constant vanishing order. This is then equivalent to $p^*(\phi_{\pi_{\mathrm{lin}}})$ having constant vanishing order, which by Corollary 4.6.3, is equivalent to the second condition in the theorem. □

4.6.3 Blowup of a bundle of linear Poisson structures

In the previous section, we have reduced the problem of blowing up invariant submanifolds from twisted Dirac structures to bundles of linear Poisson structures, which we will discuss here. The next theorem, combined with Theorem 4.6.6, yields Theorem 4.6.1.

Theorem 4.6.7 *Let* $(\underline{\mathfrak{g}}^*, \pi_{\mathrm{lin}})$ *be a bundle of linear Poisson structures corresponding to the bundle of Lie algebras* $(\underline{\mathfrak{g}}, [\,\cdot\,,\,\cdot\,])$ *over* N*. The following are equivalent.*

- *The Dirac structure* $\operatorname{graph}(\pi_{\mathrm{lin}})$ *lifts to* $\operatorname{Blup}(\underline{\mathfrak{g}}^*, 0_N)$*.*
- *The Lie algebras* $\underline{\mathfrak{g}}\big|_q$*,* $q \in N$*, have all the same constant height* k*.*

The first step in the proof is to relate the vanishing orders of π_{lin} to those of its restriction to the fibres.

Lemma 4.6.8 *Let* $(\underline{\mathfrak{g}}^*, \pi_{\mathrm{lin}})$ *be a bundle of linear Poisson structures corresponding to the bundle of Lie algebras* $(\underline{\mathfrak{g}}, [\,\cdot\,,\,\cdot\,])$ *over* N*. Let* $\ell \geq 0$*. The following are equivalent.*

- $p^*(\exp^{\pi_{\mathrm{lin}}}\lambda)$ *has constant vanishing order* ℓ *along* $\mathbb{P}(\underline{\mathfrak{g}}^*)$*, for any (local) volume form* λ *on* $\underline{\mathfrak{g}}^*$*.*
- $p_q^*(\exp^{\pi_{\mathrm{lin}}|_{\underline{\mathfrak{g}}^*|_q}}\lambda_q)$ *has constant vanishing order* ℓ *along* $\mathbb{P}(\underline{\mathfrak{g}}^*\big|_q)$*, for any volume form* λ_q *on* $\underline{\mathfrak{g}}^*\big|_q$ *and all* $q \in N$*.*

Here, $p_q\colon \operatorname{Blup}(\underline{\mathfrak{g}}^*\big|_q, \{0_q\}) \to \underline{\mathfrak{g}}^*\big|_q$ *is the blowdown map of the fibre over* q*, which can be seen as the restriction of the blowdown map* $p\colon \operatorname{Blup}(\underline{\mathfrak{g}}^*, N) \to \underline{\mathfrak{g}}^*$ *to* $p^{-1}(\underline{\mathfrak{g}}^*\big|_q)$*.*

Proof. The statements are local on N and do not depend on the chosen volume forms. So we may assume that $\underline{\mathfrak{g}}^* = \mathbb{R}^m \times N$, and choose a product volume form $\lambda = \lambda_{\mathbb{R}^m} \wedge \lambda_N$. For any $q \in N$ we have that $\pi_{\mathrm{lin}}\big|_{\mathbb{R}^m \times \{q\}}$ is a linear Poisson structure π_q on $\mathbb{R}^m$. With this, we have that

$$p^*(\exp^{\pi_{\mathrm{lin}}}\lambda)\big|_{\mathbb{R}^m \times \{q\}} = p_q^*(\exp^{\pi_q}\lambda_{\mathbb{R}^m}) \wedge \lambda_N. \tag{4.9}$$

If $p^*(\exp^{\pi_{\mathrm{lin}}}\lambda)$ has constant vanishing order ℓ along $\mathbb{P}(\underline{\mathfrak{g}}^*) = \mathbb{P}(\mathbb{R}^m) \times N$, then

$$\tilde{x}_i^{-\ell} p^*(\exp^{\pi_{\mathrm{lin}}}\lambda)\big|_{U_i \times N}$$

is smooth and nowhere vanishing, where $(U_i, (\tilde{x}_1, \ldots, \tilde{x}_m))$ is a standard chart on $\operatorname{Blup}(\mathbb{R}^m, \{0\})$ with $\mathbb{P}(\mathbb{R}^m) \cap U_i = \{\tilde{x}_i = 0\}$. Fix $q \in N$. Restricting to $\mathbb{R}^m \times \{q\}$, (4.9) implies that

$$\tilde{x}_i^{-\ell} p_q^*(\exp^{\pi_q}\lambda_{\mathbb{R}^m})\big|_{U_i}$$

is smooth and nowhere vanishing. Hence, $p_q^*(\exp^{\pi_q}\lambda_{\mathbb{R}^m})$ has constant vanishing order ℓ along $\mathbb{P}(\mathbb{R}^m)$.

The converse is proven in the same way: if $\tilde{x}_i^{-\ell} p_q^*(\exp^{\pi_q} \lambda_{\mathbb{R}^m})\big|_{U_i}$ is smooth and nowhere vanishing for all $q \in N$, then, by (4.9), also $\tilde{x}_i^{-\ell} p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{U_i \times N}$ is smooth and nowhere vanishing. □

Next, for the dual of a Lie algebra, we relate the height to the vanishing order.

Lemma 4.6.9 *Let $\mathfrak{g}$ be a Lie algebra and denote the linear Poisson structure on $\mathfrak{g}^*$ by π_{lin}. Fix a volume form λ on $\mathfrak{g}^*$. For $\xi \in \mathfrak{g}^* \setminus \{0\}$, consider the one-dimensional vector space*

$$\Sigma_{[\xi]} = (\mathbb{R}\backslash\{0\}) \cdot \xi \, \cup \, \{[\xi]\} \subseteq \mathrm{Blup}(\mathfrak{g}^*, \{0\}).$$

The vanishing order of $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{\Sigma_{[\xi]}}$ at $[\xi] \in \mathbb{P}(\mathfrak{g}^*)$ is $\dim \mathfrak{g} - 1 - \mathrm{height}(\xi)$.

Regarding the blowup as the tautological line bundle over the projective space:

$$\sigma \colon \mathrm{Blup}(\mathfrak{g}^*, \{0\}) \to \mathbb{P}(\mathfrak{g}^*),$$

the line $\Sigma_{[\xi]} = \sigma^{-1}([\xi])$ is just the fibre over $[\xi]$.

Proof of Lemma 4.6.9. Fix $\xi \in \mathfrak{g}^* \setminus \{0\}$. Let $b_1, \ldots, b_d$ be a basis of $\mathfrak{g}$ such that $\xi(b_1) = 1$ and $\xi(b_i) = 0$, for $2 \leq i \leq d$. Let π_{ij}^k denote the corresponding structure constants of $\mathfrak{g}$, i.e.

$$[b_i, b_j] = \sum_k \pi_{ij}^k \, b_k.$$

The height of ξ is half the rank of the skew-symmetric matrix

$$M_1 := (\pi_{ij}^1)_{i,j \neq 1} = (\xi[b_i, b_j])_{i,j\neq 1}.$$

Indeed, the height of ξ is precisely half the rank of the skew-symmetric bilinear form $\mathrm{d}_{\mathfrak{g}}\xi$ restricted to $\ker \xi$. In the basis $\{b_i\}_{i=2}^d$ this form is represented by the matrix $(\mathrm{d}_{\mathfrak{g}}\xi)(b_i, b_j) = -\xi([b_i, b_j])$.

Let $\{x_i\}_{i=1}^d$ denote the linear coordinates on $\mathfrak{g}^*$ induced by the basis $\{b_i\}_{i=1}^d$. In these coordinates, the Poisson structure is given by

$$\pi_{\mathrm{lin}} = \frac{1}{2} \sum_{i,j,k} x_k \pi_{ij}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Consider the chart $(U_1, \tilde{x})$ from (4.2) on $\mathrm{Blup}(\mathfrak{g}^*, \{0\})$. Using that $p^* x_1 = \tilde{x}_1$, $p^* x_j = \tilde{x}_1 \tilde{x}_j$ for $j \neq 1$, and (4.3), we see that the coefficient functions of $\tilde{\pi} = p^* \pi_{\mathrm{lin}}\big|_{U_1 \backslash \mathbb{P}(\mathfrak{g}^*)}$ are given by

$$\frac{1}{2}\tilde{\pi}_{ij} = \frac{1}{2\tilde{x}_1}\left[\pi_{ij}^1 - \tilde{x}_j \pi_{i1}^1 - \tilde{x}_i \pi_{1j}^1 + \sum_{k \neq 1} \tilde{x}_k \left(\pi_{ij}^k - \tilde{x}_j \pi_{i1}^k - \tilde{x}_i \pi_{1j}^k\right)\right] \text{ if } i, j \neq 1,$$

and

$$\frac{1}{2}\tilde{\pi}_{ij} = \frac{1}{2}(\pi_{ij}^1 + \sum_{k\neq 1} \tilde{x}_k \pi_{ij}^k) \text{ if } i = 1 \text{ or } j = 1.$$

Note that the line $\Sigma_{[\xi]}$ is included in U_1, and it is given by $\Sigma_{[\xi]} = \{\tilde{x}_2 = \ldots = \tilde{x}_d = 0\}$. Along this line, the coefficient functions simplify to

$$\tilde{\pi}_{ij} = \begin{cases} \frac{1}{\tilde{x}_1}\pi^1_{ij} & \text{if } i, j \neq 1 \\ \pi^1_{ij} & \text{if } i = 1 \text{ or } j = 1. \end{cases}$$

Then we can write

$$\tilde{\pi} = \frac{1}{\tilde{x}_1}\tilde{M}_1 + \frac{\partial}{\partial \tilde{x}_1} \wedge v, \quad \text{where } v = \sum_{j=2}^{d} \pi^1_{1,j} \frac{\partial}{\partial \tilde{x}_j}, \tag{4.10}$$

and $\tilde{M}_1$ is the matrix M_1 viewed as a constant bivector field.

Let $\lambda = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. We calculate the restriction to $\Sigma_{[\xi]}$ of the pulled back spinor.

$$\begin{aligned} p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{\Sigma_{[\xi]}} &= \exp^{p^*(\pi_{\mathrm{lin}})} p^*\lambda\big|_{\Sigma_{[\xi]}} \\ &= \tilde{x}_1^{d-1} \exp^{\tilde{\pi}} \mathrm{d}\tilde{x}_1 \wedge \cdots \wedge \mathrm{d}\tilde{x}_d \\ &= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \frac{1}{k!} \tilde{x}_1^{d-1-k} (\mathrm{i}_{\tilde{x}_1 \tilde{\pi}})^k \mathrm{d}\tilde{x}_1 \wedge \cdots \wedge \mathrm{d}\tilde{x}_d \\ &= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \frac{1}{k!} \tilde{x}_1^{d-1-k} (\mathrm{i}_v + \mathrm{d}\tilde{x}_1 \wedge)(\mathrm{i}_{\tilde{M}_1})^k \mathrm{d}\tilde{x}_2 \wedge \cdots \wedge \mathrm{d}\tilde{x}_d, \end{aligned}$$

where in the last step, we used (4.10). In this sum, each power $\tilde{x}_1^{d-1-k}$ is multiplied with a constant form, which is non-zero precisely when $(\mathrm{i}_{\tilde{M}_1})^k \mathrm{d}\tilde{x}_2 \wedge \cdots \wedge \mathrm{d}\tilde{x}_d \neq 0$, which is equivalent to $\operatorname{rank}(M_1) \geq 2k$. Because $\frac{1}{2}\operatorname{rank}(M_1) = \operatorname{height}(\xi)$, the largest k such that the coefficient of $\tilde{x}_1^{d-1-k}$ in $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{\Sigma_{[\xi]}}$ is non-zero is $\operatorname{height}(\xi)$. Hence, the vanishing order is $\ell = d - 1 - \operatorname{height}(\xi)$. ☐

The next lemma concludes the proof of Theorem 4.6.7.

Lemma 4.6.10 *Let $\mathfrak{g}$ be a Lie algebra and denote the linear Poisson structure on $\mathfrak{g}^*$ by π_{lin}. Fix a volume form λ on $\mathfrak{g}^*$. Then the following are equivalent.*

- *The form $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)$ has constant vanishing order ℓ along $\mathbb{P}(\mathfrak{g}^*)$.*
- *The height of elements $\xi \in \mathfrak{g}^* \backslash \{0\}$ is a constant k.*

In this case, $\ell + k = \dim \mathfrak{g} - 1$.

Proof. In view of Lemma 4.6.9, we need to check that the following are equivalent:

- $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)$ has constant vanishing order ℓ along $\mathbb{P}(\mathfrak{g}^*)$,
- for each $[\xi] \in \mathbb{P}(\mathfrak{g}^*)$, $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{\Sigma_{[\xi]}}$ has vanishing order ℓ at $[\xi]$.

Let $(U_i, \tilde{x})$ be a standard coordinate system on $\mathrm{Blup}(\mathfrak{g}^*, \{0\})$, with $U_i \cap \mathbb{P}(\mathfrak{g}^*) = \{\tilde{x}_i = 0\}$. The first condition is equivalent to $\tilde{x}_i^{-\ell} p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)$ being smooth and nowhere vanishing on U_i. This implies that the restriction $\tilde{x}_i^{-\ell} p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{\Sigma_{[\xi]}}$ is smooth and nowhere vanishing. For any $[\xi] \in U_i$, we have that $\tilde{x}_i \colon \Sigma_{[\xi]} \to \mathbb{R}$ is a linear isomorphism. Hence, $p^*(\exp^{\pi_{\mathrm{lin}}} \lambda)\big|_{\Sigma_{[\xi]}}$ has vanishing order ℓ at $[\xi]$. The converse implication is proven in exactly the same way. ☐

4.7 Lifting Poisson structures: a geometric approach

In this section, we present a more geometric proof, which does not use spinors, of Theorem 4.6.1 for the particular case of blowing up a zero of a Poisson manifold. This implies also the case of symplectic leaves of a Poisson manifold and, more generally, of presymplectic leaves of a Dirac manifold, by using the splitting theorem of Weinstein [Wei83, Theorem 2.1] and Blohmann [Blo17], respectively. Moreover, we give alternative descriptions of the constant height condition for a Lie algebra.

Theorem 4.7.1 *Let (M, π) be a Poisson manifold and $q \in M$ a zero of π, i.e. $\pi(q) = 0$. Let $\mathfrak{g} := T_q^*M$ denote the isotropy Lie algebra of π at q. The following are equivalent.*

1. *The Dirac structure* $\mathrm{graph}(\pi)$ *lifts to a Dirac structure $\tilde{D}$ on the blowup* $\mathrm{Blup}(M, \{q\})$.
2. *There exists $k \in \mathbb{N}_0$ such that for all $v \in T_qM \setminus \{0\}$ the subspace*
$$\mathcal{D}_{[v]} := \left\{ (\widetilde{\pi^\sharp \alpha})_{[v]} : \alpha \in \Omega^1(M),\ \alpha_q(v) = 0 \right\} \subseteq T_{[v]}\mathbb{P}(T_qM) \tag{4.11}$$
has rank $2k$. Here, $\tilde{X}$ denotes the lift to $\mathrm{Blup}(M, \{q\})$ *of a vector field X on M vanishing at q.*
3. *There exists $k \in \mathbb{N}_0$ such that for all $\xi \in \mathfrak{g}^* \setminus \{0\}$, the coadjoint orbit $\mathcal{O}_\xi$ through ξ satisfies*
$$\dim(\mathcal{O}_\xi) = \begin{cases} 2k+2 & \textit{if } T_\xi\mathcal{O}_\xi \textit{ contains the radial line } \mathbb{R}\xi, \\ 2k & \textit{otherwise.} \end{cases} \tag{4.12}$$
4. *The Lie algebra $\mathfrak{g}$ has constant height k, i.e. for all $\xi \in \mathfrak{g}^* \setminus \{0\}$ we have*
$$\xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^k \neq 0 \quad \textit{and} \quad \xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} = 0, \tag{4.13}$$
where $\mathrm{d}_{\mathfrak{g}} \colon \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+1}\mathfrak{g}^*$ denotes the Chevalley-Eilenberg differential.

If any (and thus all) conditions are satisfied, the blowdown map is a forward Dirac map, and
$$\tilde{D} \cap T\mathbb{P}(T_qM) = \mathcal{D}.$$
Moreover, $\tilde{D}$ is the graph of a Poisson structure if and only if $k = 0$.

Remark 4.7.2 The integers k appearing in conditions 2,3 and 4 all agree.

Remark 4.7.3 The space D appearing in condition 2 depends only on the linearisation π_{lin} of π at q. Indeed, in a chart centred at q, the vector fields $(\pi - \pi_{\mathrm{lin}})^\sharp\alpha$ vanish to second order at 0, and so their lift to $\mathrm{Blup}(M, \{q\})$ vanishes along $\mathbb{P}(T_qM)$.

4.7.1 Examples for Theorem 4.7.1

Before proving Theorem 4.7.1, we present some explicit examples.

Example 4.7.4 We check explicitly the conditions of Theorem 4.7.1 for the Poisson manifold $\mathfrak{so}(3)^*$, with the linear Poisson structure π, showing that it lifts to a Dirac structure on the blowup which is not a Poisson structure, and describing its geometry. We start from condition 4.

4. The Lie algebra $\mathfrak{so}(3)$ admits a basis $\{X_1, X_2, X_3\}$ such that $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$. Denote by $\{\theta_1, \theta_2, \theta_3\}$ the dual basis. For all $\xi \in \mathfrak{so}(3)^* \setminus \{0\}$ we have

   $$\xi \wedge (\mathrm{d}_{\mathfrak{so}(3)}\xi) = -\Big(\sum_{j=1}^{3}(\xi(X_j))^2\Big)\theta_1 \wedge \theta_2 \wedge \theta_3 \neq 0,$$

   while $\xi \wedge (\mathrm{d}_{\mathfrak{so}(3)}\xi)^2 = 0$ for dimensional reasons. Hence, condition 4 is satisfied for $k = 1$.

3. The coadjoint orbits (apart from the origin) are concentric spheres centred around the origin. In particular, they have dimension 2, and they do not contain any radial line. Therefore, condition 3 is satisfied for $k = 1$.

1. Denote by $\{x_i\}$ the linear coordinates on $\mathfrak{so}(3)^*$ arising from the basis $\{X_i\}$. A frame for $\mathrm{graph}(\pi)$ is given by $\{(\pi^\sharp \mathrm{d}x_i, \mathrm{d}x_i)\}$, where for instance $\pi^\sharp \mathrm{d}x_1 = x_3\partial_2 - x_2\partial_3$. In the chart U_1 of Section 4.2 one has that $(\pi^\sharp \mathrm{d}x_i, p^*\mathrm{d}x_i)$ equals, for $i = 1, 2, 3$ respectively,

   $$\begin{pmatrix} \tilde{x}_3\tilde{\partial}_2 - \tilde{x}_2\tilde{\partial}_3 \\ \mathrm{d}\tilde{x}_1 \end{pmatrix}, \quad \begin{pmatrix} -\tilde{x}_1\tilde{x}_3\tilde{\partial}_1 + \tilde{x}_2\tilde{x}_3\tilde{\partial}_2 + (1+\tilde{x}_3^2)\tilde{\partial}_3 \\ \tilde{x}_1\mathrm{d}\tilde{x}_2 + \tilde{x}_2\mathrm{d}\tilde{x}_1 \end{pmatrix}, \quad \text{and}$$

   $$\begin{pmatrix} \tilde{x}_1\tilde{x}_2\tilde{\partial}_1 - (1+\tilde{x}_2^2)\tilde{\partial}_2 - \tilde{x}_2\tilde{x}_3\tilde{\partial}_3 \\ \tilde{x}_1\mathrm{d}\tilde{x}_3 + \tilde{x}_3\mathrm{d}\tilde{x}_1 \end{pmatrix}.$$

   This can be computed using (4.2) and (4.3); here we use the notation $\tilde{\partial}_j := \partial_{\tilde{x}_j}$. We know that these three sections are linearly independent away from $\mathbb{P}(\mathfrak{so}(3)^*) \cap U_1 = \{\tilde{x}_1 = 0\}$, but they are also on $\{\tilde{x}_1 = 0\}$, since there the determinant of the coefficients of $\tilde{\partial}_2, \tilde{\partial}_3, \mathrm{d}\tilde{x}_1$ is given by $(1+(\tilde{x}_2)^2+(\tilde{x}_3)^2)^2 > 0$. Hence, these three sections span a Dirac structure on U_1 that lifts π. A similar computation can be done for the charts U_2 and U_3, showing directly that π lifts to a Dirac structure $\tilde{D}$.

2. Let $v = (v_1, v_2, v_3) \in \mathfrak{so}(3)^* \backslash \{0\}$. Without loss of generality, assume that $v_1 \neq 0$. Then $[v]$ is covered by the chart $(U_1, \tilde{x})$, and in these coordinates $(\tilde{x}_1([v]), \tilde{x}_2([v]), \tilde{x}_3([v])) = (0, \frac{v_2}{v_1}, \frac{v_3}{v_1})$. A basis for the covectors that annihilate v is given by

$$\alpha_2 := v_3 \mathrm{d}x_1 - v_1 \mathrm{d}x_3, \quad \alpha_3 := v_1 \mathrm{d}x_2 - v_2 \mathrm{d}x_1,$$

which we regard also as constant 1-forms. Using the above expressions for $\widetilde{\pi^\sharp \mathrm{d}x_i}$, we obtain

$$(\widetilde{\pi^\sharp \alpha_2})_{[v]} = \frac{v_1^2 + v_2^2 + v_3^2}{v_1} \tilde{\partial}_2, \qquad (\widetilde{\pi^\sharp \alpha_3})_{[v]} = \frac{v_1^2 + v_2^2 + v_3^2}{v_1} \tilde{\partial}_3.$$

Hence, we obtain that

$$\mathcal{D}_{[v]} = \operatorname{span}\{\tilde{\partial}_2, \tilde{\partial}_3\} = T_{[v]}\mathbb{P}(\mathfrak{so}(3)^*).$$

Therefore, $\mathcal{D} = T\mathbb{P}(\mathfrak{so}(3)^*)$, and so the rank of $\mathcal{D}$ is two everywhere. So 2. holds for $k = 1$.

Further, it is easy to see that on $\mathbb{P}(\mathfrak{so}(3)^*) \cap U_1 = \{\tilde{x}_1 = 0\}$ the Dirac structure $\tilde{D}$ is spanned also by $(\tilde{\partial}_2, 0), (\tilde{\partial}_3, 0), (0, \mathrm{d}\tilde{x}_1)$. This implies that $\mathbb{P}(\mathfrak{so}(3)^*)$ together with the zero 2-form is a presymplectic leaf of $\tilde{D}$. In particular, the Dirac structure $\tilde{D}$ is regular.

Example 4.7.5 We briefly look at the conditions of Theorem 4.7.1 for the Poisson manifold $(\mathfrak{sl}_2(\mathbb{R})^*, \pi)$, building on Example 4.7.4, and showing that the Poisson structure does not lift.

4. For all $\xi \in \mathfrak{sl}_2(\mathbb{R})^* \setminus \{0\}$ we have

$$\xi \wedge (\mathrm{d}_{\mathfrak{sl}_2(\mathbb{R})}\xi) = \big(-(\xi(e_1))^2 - (\xi(e_2))^2 + (\xi(e_3))^2 \big)\, e_1^* \wedge e_2^* \wedge e_3^*,$$

which vanishes exactly on the two coadjoint orbits given by half-cones.

3. The coadjoint orbits (apart from the origin) are given by hyperboloids, paraboloids and two half-cones. They all have dimension 2, but the half-cones contain the radial line, while the other orbits do not.

4.7.2 Proof of the equivalence $1 \Leftrightarrow 2$ in Theorem 4.7.1

We prove now the equivalence $1 \Leftrightarrow 2$, together with the two final statements of Theorem 4.7.1.

We may assume that $M = \mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$, and that $q = 0$. We use the chart $(U_n, \tilde{x})$ of $\mathrm{Blup}(\mathbb{R}^n, \{0\})$ from Section 4.2, in which $U_n \cap \mathbb{P}(\mathbb{R}^n) = \{\tilde{x}_n = 0\}$.

Note that for all $i < n$ we have

$$p^*\mathrm{d}x_i - \tilde{x}_i p^* \mathrm{d}x_n = \tilde{x}_n \mathrm{d}\tilde{x}_i + \tilde{x}_i \mathrm{d}\tilde{x}_n - \tilde{x}_i \mathrm{d}\tilde{x}_n = \tilde{x}_n \mathrm{d}\tilde{x}_i. \tag{4.14}$$

We consider the sections

$$\left\{ \begin{pmatrix} \widetilde{\pi^\sharp \mathrm{d}x_i} \\ p^* \mathrm{d}x_i \end{pmatrix} \right\}_{i \leq n}$$

of $\mathbb{T}\mathrm{Blup}(\mathbb{R}^n, \{0\})$, which away from $\mathbb{P}(\mathbb{R}^n)$ are a frame for the Dirac structure corresponding to $\mathrm{graph}(\pi)$ under the blowdown map p. Over U_n, for $i < n$, replace the i-th section by itself minus $\tilde{x}_i$ times the n-th section. This yields another family of sections with the same property,

$$e_n = \begin{pmatrix} \widetilde{\pi^\sharp \mathrm{d}x_n} \\ \mathrm{d}\tilde{x}_n \end{pmatrix} \quad \text{and} \quad e_i = \begin{pmatrix} \widetilde{\pi^\sharp \mathrm{d}x_i} - \tilde{x}_i \widetilde{\pi^\sharp \mathrm{d}x_n} \\ \tilde{x}_n \mathrm{d}\tilde{x}_i \end{pmatrix} \quad \text{for } i < n, \tag{4.15}$$

where we used (4.14) for the cotangent part. Notice that the vector component is tangent to $\mathbb{P}(\mathbb{R}^n)$ by Lemma 4.2.1, since π vanishes at the origin, and for $i < n$ the covector components vanish on $\mathbb{P}(\mathbb{R}^n)$.

Lemma 4.7.6 *For any $[v] \in \mathbb{P}(\mathbb{R}^n) \cap U_n$, the span of the tangent components of the $\{e_i\}_{i<n}$ is precisely the subspace $\mathcal{D}_{[v]}$ given in* (4.11).

Proof. Indeed, fix $[v] \in U_n \cap \mathbb{P}(\mathbb{R}^n)$ and set $\alpha_i^{[v]} := \mathrm{d}x_i - \tilde{x}_i([v])\mathrm{d}x_n$. Then

$$\left(\widetilde{\pi^\sharp \mathrm{d}x_i} - \tilde{x}_i \widetilde{\pi^\sharp \mathrm{d}x_n}\right)_{[v]} = (\widetilde{\pi^\sharp \mathrm{d}x_i})_{[v]} - \tilde{x}_i([v])(\widetilde{\pi^\sharp \mathrm{d}x_n})_{[v]} = (\widetilde{\pi^\sharp \alpha_i^{[v]}})_{[v]}.$$

The key observation is that $\alpha_i^{[v]}$ annihilates $v \in T_0\mathbb{R}^n$, as $\tilde{x}_i([v]) = \frac{v_i}{v_n}$. For $i < n$ these covectors form a basis of the annihilator $(\mathbb{R}v)^{\mathrm{ann}}$, proving the claim. ☐

Proof of the implication $1 \Leftarrow 2$ in Theorem 4.7.1

Assume the constant rank condition in the statement, i.e. the subspaces $\mathcal{D}_{[v]}$, $[v] \in \mathbb{P}(\mathbb{R}^n)$, have constant rank $2k$.

Let $[v_0] \in \mathbb{P}(\mathbb{R}^n) \cap U_n$. Using Lemma 4.7.6, after restricting to a neighbourhood $U' \subseteq U_n$ of $[v_0]$, by permuting the $\{e_i\}_{1 \leq i \leq n-1}$, we may assume that

- the sections $\{e_i\}_{1 \leq i \leq 2k}$ restricted to $\mathbb{P}(\mathbb{R}^n) \cap U'$ are a frame for the distribution $\mathcal{D}\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'}$.

The other sections can be written along $\mathbb{P}(\mathbb{R}^n) \cap U'$ as unique linear combinations:

$$e_j\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'} = \sum_{i=1}^{2k} a_j^i e_i\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'}, \quad 2k+1 \leq j \leq n-1.$$

Extend the smooth functions a_j^i to U' (and denote these in the same way), and define the sections

$$e_j' := e_j - \sum_{i=1}^{2k} a_j^i e_i, \quad 2k+1 \leq j \leq n-1.$$

We obtain sections of $\mathbb{T}U'$

$$e_1, \dots, e_{2k}, e'_{2k+1}, \dots, e'_{n-1}, e_n,$$

with the following properties:

- on $U' \setminus \mathbb{P}(\mathbb{R}^n)$ they form a frame for the lifted Dirac structure,
- the $\{e'_j\}_{2k+1 \leq j \leq n-1}$ vanish on $\mathbb{P}(\mathbb{R}^n) \cap U'$.

Since $\tilde{x}_n$ is a defining function for $\mathbb{P}(\mathbb{R}^n)$ inside U_n, we can divide the e'_j by $\tilde{x}_n$ to obtain a smooth section of $\mathbb{T}U'$, and obtain sections

$$e_1, \dots, e_{2k}, \frac{1}{\tilde{x}_n} e'_{2k+1}, \cdots, \frac{1}{\tilde{x}_n} e'_{n-1}, e_n \tag{4.16}$$

which on $U' \backslash \mathbb{P}(\mathbb{R}^n)$ are a frame for the lifted Dirac structure. These sections (4.16) are pointwise linearly independent also at points of $\mathbb{P}(\mathbb{R}^n) \cap U'$. Indeed, using the explicit form (4.15), along $\mathbb{P}(\mathbb{R}^n) \cap U'$ these elements have the following properties:

- $e_i\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'}$, for $1 \leq i \leq 2k$, have only vector components and are linearly independent;
- $\frac{1}{\tilde{x}_n} e'_j\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'}$, $2k+1 \leq j \leq n-1$, has cotangent components

$$\Big(\mathrm{d}\tilde{x}_j - \sum_{i=1}^{2k} a^i_j \mathrm{d}\tilde{x}_i\Big)\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'};$$

- e_n has cotangent component $\mathrm{d}\tilde{x}_n\big|_{\mathbb{P}(\mathbb{R}^n) \cap U'}$.

By covering $\mathbb{P}(\mathbb{R}^n)$ with such open sets U', we obtain that the vector subbundle

$$p^*(\mathrm{graph}(\pi)\big|_{\mathbb{R}^n \setminus \{0\}}) \subseteq \mathbb{T}\mathrm{Blup}(\mathbb{R}^n, \{0\})\big|_{\mathrm{Blup}(\mathbb{R}^n, \{0\}) \setminus \mathbb{P}(\mathbb{R}^n)}$$

extends to a smooth vector subbundle of $\mathbb{T}\mathrm{Blup}(\mathbb{R}^n, \{0\})$. By [Blo17, Remark 2.9.], the extension is a Dirac structure, as all axioms hold on the dense open subset $\mathrm{Blup}(\mathbb{R}^n, \{0\}) \setminus \mathbb{P}(\mathbb{R}^n)$.

It is a good moment to prove one of the additional, final statements of Theorem 4.7.1.

Lemma 4.7.7 *The pullback Dirac structure is Poisson if and only if* $k = 0$.

Proof. For the frame 4.16 of the lift $\tilde{D}$, the cotangent components of the first $2k$-elements vanish along $\mathbb{P}(\mathbb{R}^n)$ and the cotangent components of the last $n - 2k$ are linearly independent along $\mathbb{P}(\mathbb{R}^n)$. This proves the claim. ☐

Proof of the implication $1 \Rightarrow 2$ in Theorem 4.7.1

Assume that graph(π) lifts to a Dirac structure $\tilde{D}$ on Blup($\mathbb{R}^m, \{0\}$).

The crucial observation for the proof of the implication $1 \Rightarrow 2$ is the following equality, asserted in Theorem 4.7.1:

$$\tilde{D}\big|_{\mathbb{P}(\mathbb{R}^n)} \cap T\mathbb{P}(\mathbb{R}^n) = \mathcal{D}. \tag{4.17}$$

Note that the rank of the left-hand side (the intersection of two subbundles) can locally only decrease, and the rank of the right-hand side (the image of a vector bundle map) can locally only increase. Therefore, their equality implies 2 in Theorem 4.7.1: $\mathcal{D}$ has constant rank.

To prove (4.17), we start with a technical result.

Lemma 4.7.8 *Let $\gamma \in \Omega^1(\mathrm{Blup}(\mathbb{R}^n, \{0\}))$. If γ vanishes at $[v] \in \mathbb{P}(\mathbb{R}^n)$, then there exists $\beta \in \Omega^1(\mathbb{R}^n)$ such that $p^*(\beta_{tv}) = \gamma_{p^{-1}(tv)}$ for all $t \neq 0$. Moreover, $\beta_0(v) = 0$.*

Proof. Without loss of generality, we may assume that $v_n \neq 0$, where $v = (v_1, \ldots, v_n)$. We will use coordinates $\tilde{x}_1, \ldots, \tilde{x}_n$ on $U_n \subseteq \mathrm{Blup}(\mathbb{R}^n, \{0\})$ as in Section 4.2. The blowdown map p satisfies the following, as one sees using (4.2):

$$p^*(\mathrm{d}x_n) = \mathrm{d}\tilde{x}_n \quad \text{and} \quad p^*\Big(\frac{\mathrm{d}x_i}{x_n} - \frac{x_i}{x_n^2}\mathrm{d}x_n\Big) = \mathrm{d}\tilde{x}_i \text{ for all } 1 \leq i \leq n-1.$$

On the domain U_n of the chart, write γ as $\sum_{i=1}^n f_i \mathrm{d}\tilde{x}_i$ for smooth functions f_i. Consider the line

$$\Sigma_{[v]} := p^{-1}(\mathbb{R}v \setminus \{0\}) \cup [v] \subseteq \mathrm{Blup}(\mathbb{R}^n, \{0\}). \tag{4.18}$$

The blowdown map gives an isomorphism $\Sigma_{[v]} \cong \mathbb{R}v$. Since γ vanishes at $[v]$, on $\Sigma_{[v]}$ we can write $f_i = \tilde{x}_n \widehat{f_i} \circ p$ for all i, for smooth functions $\widehat{f_i}$ on $\mathbb{R} \cdot v$. We extend these functions to smooth functions on $\mathbb{R}^n$, which we denote in the same way. Using (4.2), for $t \neq 0$ we have

$$\begin{aligned}
\gamma\big|_{p^{-1}(tv)} &= p^*\Big(\sum_{i<n} x_n \widehat{f_i}(x)\Big(\frac{\mathrm{d}x_i}{x_n} - \frac{x_i}{x_n^2}\mathrm{d}x_n\Big) + x_n \widehat{f_n}(x)\mathrm{d}x_n\Big)\Big|_{p^{-1}(tv)} \\
&= p^*\left(\sum_{i<n} \widehat{f_i}(tv)\mathrm{d}x_i\big|_{tv} + \Big(tv_n \widehat{f_n}(tv) - \sum_{i<n} \widehat{f_i}(tv)\frac{v_i}{v_n}\Big)\mathrm{d}x_n\big|_{tv}\right).
\end{aligned}$$

This expression extends smoothly to $t = 0$, and yields an explicit formula:

$$\beta := \sum_{i<n} \widehat{f_i}(x)\mathrm{d}x_i + \Big(x_n \widehat{f_n}(x) - \sum_{i<n} \widehat{f_i}(x)\frac{v_i}{v_n}\Big)\mathrm{d}x_n.$$

At $t = 0$, we have that:

$$\beta_0(v) = \sum_{i<n} \widehat{f_i}(0)v_i + \Big(0 - \sum_{i<n} \widehat{f_i}(0)\frac{v_i}{v_n}\Big)v_n = 0.$$

□

Proof of Equation (4.17). We just need to prove the inclusion "$\subseteq$" in (4.17), since the other inclusion follows from Lemma 4.7.6 and the lines preceding it. Let $[v] \in \mathbb{P}(\mathbb{R}^n)$ and let $X_{[v]} \in \tilde{D}_{[v]} \cap T_{[v]}\mathbb{P}(\mathbb{R}^n)$. Extend $(X_{[v]}, 0)$ to a section $(X, \gamma) \in \Gamma(\tilde{D})$. Then there exists a unique $\alpha \in \Omega^1(\mathbb{R}^n \setminus \{0\})$ such that

$$\begin{pmatrix} X \\ \gamma \end{pmatrix}\Big|_{\mathrm{Blup}(\mathbb{R}^n,\{0\})\setminus\mathbb{P}(\mathbb{R}^n)} = \widetilde{\begin{pmatrix} \pi^\sharp\alpha \\ p^*\alpha \end{pmatrix}}\Big|_{\mathrm{Blup}(\mathbb{R}^n,\{0\})\setminus\mathbb{P}(\mathbb{R}^n)},$$

since the blowdown map p is a diffeomorphism away from $\mathbb{P}(\mathbb{R}^n)$. Lemma 4.7.8 implies that there exists $\beta \in \Omega^1(\mathbb{R}^n)$ which agrees with α on the line $\mathbb{R}v \setminus \{0\}$. We consider a limit along the line $\Sigma_{[v]}$ (4.18). We have

$$X_{[v]} = \lim_{t\to 0} \widetilde{\pi^\sharp\alpha}\big|_{p^{-1}(tv)} = \lim_{t\to 0} \widetilde{\pi^\sharp\beta}\big|_{p^{-1}(tv)} = \widetilde{\pi^\sharp\beta}\big|_{[v]} = \widetilde{\pi^\sharp\beta_0}\big|_{[v]},$$

where β_0 is viewed as a constant 1-form, and in the last equality uses that $\pi^\sharp(\beta - \beta_0)$ vanishes to second order at 0, so its lift is zero along $\mathbb{P}(\mathbb{R}^n)$. By Lemma 4.7.8, $\beta_0(v) = 0$. Hence, $X_{[v]} \in \mathcal{D}_{[v]}$. $\square$

Finally, we prove also the following assertion of Theorem 4.7.1.

Lemma 4.7.9 *The blowdown map* $p\colon \mathrm{Blup}(\mathbb{R}^n, \{0\}) \to \mathbb{R}^n$ *is a forward Dirac map.*

Proof. We need to show that $\mathfrak{F}_p(\tilde{D}_{[v]}) = T_0^*\mathbb{R}^n$, for all $[v] \in \mathbb{P}(\mathbb{R}^n)$. Let $\alpha_0 \in T_0^*\mathbb{R}^n$. We regard α_0 as a constant 1-form. Then $(\widetilde{\pi^\sharp\alpha_0}, p^*\alpha_0)$ belongs to $\tilde{D}$ outside of $\mathbb{P}(\mathbb{R}^n)$. Since $\tilde{D}$ is a closed subset, also $(\widetilde{\pi^\sharp\alpha_0}, p^*\alpha_0)_{[v]} \in \tilde{D}_{[v]}$, for all $[v] \in \mathbb{P}(\mathbb{R}^n)$. Since $T_{[v]}p((\widetilde{\pi^\sharp\alpha_0})_{[v]}) = 0$, $\alpha_0 \in \mathfrak{F}_p(\tilde{D}_{[v]})$. $\square$

4.7.3 The equivalence $2 \Leftrightarrow 3$ in Theorem 4.7.1

We now give a more explicit description of the singular distribution $\mathcal{D}$ in (4.11), solely in terms of the coadjoint orbits of the Lie algebra $\mathfrak{g}$.

Lemma 4.7.10 *Let* $\mathfrak{g}$ *be a Lie algebra. Let* $\xi \in \mathfrak{g}^*$ *be a non-zero element, and denote by* $\mathcal{O}_\xi$ *the coadjoint orbit through* ξ. *Denote by* π_{lin} *the linear Poisson structure on* $\mathfrak{g}^*$, *hence* $\mathrm{im}\,((\pi_{\mathrm{lin}})^\sharp_\xi) = T_\xi\mathcal{O}_\xi$. *Then*

i) $T_\xi\mathcal{O}_\xi$ *contains the radial line* $\mathbb{R}\xi \Leftrightarrow (\pi_{\mathrm{lin}})^\sharp_\xi((\mathbb{R}\xi)^{\mathrm{ann}}) \subsetneq T_\xi\mathcal{O}_\xi$,

and in this case, we have that

ii) $\mathbb{R}\xi \subseteq (\pi_{\mathrm{lin}})^\sharp_\xi((\mathbb{R}\xi)^{\mathrm{ann}})$.

Proof. Since $B := (\pi_{\mathrm{lin}})^\sharp_\xi$ is skew-symmetric, $(\mathrm{im}\,B)^{\mathrm{ann}} = \ker B$. So, we obtain i):

$$\xi \in \mathrm{im}\,B \;\Leftrightarrow\; \mathbb{R}\xi \cap \mathrm{im}\,B \neq \{0\} \;\Leftrightarrow\; (\mathbb{R}\xi)^{\mathrm{ann}} + \ker B \neq \mathfrak{g} \;\Leftrightarrow\; B((\mathbb{R}\xi)^{\mathrm{ann}}) \neq \mathrm{im}\,B,$$

where the last equivalence follows from the rank-nullity theorem. In this case, there is $X \in \mathfrak{g}$ such that $B(X) = \xi$. By skew-symmetry, we get $0 = \langle BX, X\rangle = \langle \xi, X\rangle$, i.e. $X \in (\mathbb{R}\xi)^{\mathrm{ann}}$. $\square$

From Lemma 4.7.10 we obtain the equivalence $2 \Leftrightarrow 3$ in Theorem 4.7.1.

Corollary 4.7.11 *Let (M, π) be Poisson and $q \in M$ with $\pi(q) = 0$, consider the Lie algebra $\mathfrak{g} = T_q^*M$. For every non-zero $\xi \in \mathfrak{g}^*$, denote by $\mathcal{O}_\xi$ the coadjoint orbit through ξ. Then the dimension of the subspace $\mathcal{D}_{[\xi]}$ of $T_{[\xi]}(\mathbb{P}(T_qM))$ of* (4.11) *reads:*

$$\dim(\mathcal{D}_{[\xi]}) = \begin{cases} \dim(\mathcal{O}_\xi) - 2 & \text{if } T_\xi\mathcal{O}_\xi \text{ contains the radial line } \mathbb{R}\xi, \\ \dim(\mathcal{O}_\xi) & \text{otherwise.} \end{cases}$$

Proof. The subspace $\{(\pi_{\mathrm{lin}})^\sharp_\xi \alpha : \alpha \in (\mathbb{R}\xi)^{\mathrm{ann}}\}$ of $\mathrm{im}\,((\pi_{\mathrm{lin}})^\sharp_\xi) = T_\xi\mathcal{O}_\xi$ has codimension either zero or one. It has codimension one exactly when $T_\xi\mathcal{O}_\xi$ contains the radial line $\mathbb{R}\xi$, by Lemma 4.7.10 i). In that case, the image of the subspace under the projectivisation has one dimension less, by Lemma 4.7.10 ii). The statement follows by re-expressing $\mathcal{D}_{[\xi]}$ (4.11) by means of π_{lin} (Remark 4.7.3). ☐

4.7.4 The equivalence $3 \Leftrightarrow 4$ in Theorem 4.7.1

The following lemma concludes the proof of Theorem 4.7.1.

Lemma 4.7.12 *Let $\mathfrak{g}$ be a Lie algebra and $\xi \in \mathfrak{g}^* \backslash \{0\}$. Let $\mathcal{O}_\xi$ be the coadjoint orbit of ξ, and $k \in \mathbb{N}_0$ be the height of ξ, i.e. $\xi \wedge (\mathrm{d}_\mathfrak{g}\xi)^k \neq 0$ and $\xi \wedge (\mathrm{d}_\mathfrak{g}\xi)^{k+1} = 0$. One of the following alternatives holds.*

(1) $(\mathrm{d}_\mathfrak{g}\xi)^{k+1} = 0$, which is equivalent to (1') $\dim(\mathcal{O}_\xi) = 2k$, and also to (1") $\xi \notin T_\xi\mathcal{O}_\xi$.

(2) $(\mathrm{d}_\mathfrak{g}\xi)^{k+1} \neq 0$, which is equivalent to (2') $\dim(\mathcal{O}_\xi) = 2k+2$, and also to (2") $\xi \in T_\xi\mathcal{O}_\xi$.

Proof. First, note that we have

$$T_\xi\mathcal{O}_\xi = \big(\ker(\mathrm{d}_\mathfrak{g}\xi)\big)^{\mathrm{ann}},$$

hence,

$$\dim(\mathcal{O}_\xi) = \mathrm{rank}\,(\mathrm{d}_\mathfrak{g}\xi).$$

(1) If $(\mathrm{d}_\mathfrak{g}\xi)^{k+1} = 0$, then, since $(\mathrm{d}_\mathfrak{g}\xi)^k \neq 0$, it follows that $\dim(\mathcal{O}_\xi) = \mathrm{rank}\,(\mathrm{d}_\mathfrak{g}\xi) = 2k$. Moreover, $\xi \notin T_\xi\mathcal{O}_\xi$. Indeed, if $\xi \in \big(\ker(\mathrm{d}_\mathfrak{g}\xi)\big)^{\mathrm{ann}}$, then, since $k = \frac{1}{2}\mathrm{rank}\,\mathrm{d}_\mathfrak{g}\xi$, we would obtain $\xi \wedge (\mathrm{d}_\mathfrak{g}\xi)^k = 0$.

(2) If $(\mathrm{d}_\mathfrak{g}\xi)^{k+1} \neq 0$, then, since

$$0 = \mathrm{d}_\mathfrak{g}(0) = \mathrm{d}_\mathfrak{g}(\xi \wedge (\mathrm{d}_\mathfrak{g}\xi)^{k+1}) = (\mathrm{d}_\mathfrak{g}\xi)^{k+2},$$

we have $\dim(\mathcal{O}_\xi) = 2k+2$. By contracting the identity $\xi \wedge (\mathrm{d}_\mathfrak{g}\xi)^{k+1} = 0$ with any vector in $\ker(\mathrm{d}_\mathfrak{g}\xi)$, we see that $\ker(\mathrm{d}_\mathfrak{g}\xi) \subseteq \xi^{\mathrm{ann}}$. Taking annihilators we obtain $\xi \in T_\xi\mathcal{O}_\xi$.

Since (1') and (2') are mutually exclusive, it follows that they imply (1) and (2), respectively. The same holds for (1") and (2"). ☐

4.8 Classification of Lie algebras of constant height

In this section, we classify all Lie algebras that satisfy the conditions stated in Theorem 4.6.1.

Theorem 4.8.1 *Any Lie algebra $\mathfrak{g}$ of constant height is isomorphic to one of the following.*

- *An abelian Lie algebra $\mathbb{R}^n$—this has height* 0.
- *The semi-direct product $\mathbb{R} \ltimes \mathbb{R}^n$, for the representation $\lambda \mapsto \lambda \mathrm{id}_{\mathbb{R}^n}$—this has height* 0.
- *The Lie algebra $\mathfrak{so}(3)$—this has height* 1.

Is useful to introduce the following terminology.

Definition 4.8.2 We say that an element $\xi \in \mathfrak{g}^* \setminus \{0\}$ is of **type (1)** if it satisfies condition (1) of Lemma 4.7.12, and of **type (2)** if it satisfies condition (2) of Lemma 4.7.12.

Remark 4.8.3 The notions of **height** and **type** of an element $\xi \in \mathfrak{g}^* \setminus \{0\}$ are encoded by the notion of **Cartan class** used in [GR19, Section 2.1]. This number is characterised by (see [GR19, Section 2.1])

$$\mathrm{class}(\xi) = 2k+1 \Leftrightarrow \xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^k \neq 0 \text{ and } (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} = 0,$$
$$\mathrm{class}(\xi) = 2k+2 \Leftrightarrow (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} \neq 0 \text{ and } \xi \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} = 0.$$

The definitions and $\mathrm{d}_{\mathfrak{g}}(\alpha \wedge (\mathrm{d}_{\mathfrak{g}}\alpha)^k) = (\mathrm{d}_{\mathfrak{g}}\alpha)^{k+1}$ imply the equality:

$$\mathrm{class}(\xi) = 2 \cdot \mathrm{height}(\xi) + \mathrm{type}(\xi).$$

The notion of Cartan class was used in [GR19, Prop. 2.9] to show that $\mathfrak{so}(3)$ or $\mathfrak{sl}_2(\mathbb{R})$ are, up to isomorphism, the only Lie algebras whose nontrivial coadjoint orbits have **codimension** 1. Notice that Theorem 4.8.1 implies a variation of this statement: any **compact** Lie algebra, such that all coadjoint orbits (except for the origin) have the same dimension, must be abelian or isomorphic to $\mathfrak{so}(3)$.

An important step in the proof is to show that any semisimple Lie algebra of constant height is isomorphic to $\mathfrak{so}(3)$ (Theorem 4.8.6). In order to show compactness, we will need the following result.

Lemma 4.8.4 *For a semisimple Lie algebra $\mathfrak{g}$ of constant height, all elements in $\mathfrak{g}^* \setminus \{0\}$ are of type (1).*

Proof. Let k denote the constant height of $\mathfrak{g}$. Suppose $\xi \in \mathfrak{g}^* \setminus \{0\}$ is of type (2), i.e. $(\mathrm{d}_{\mathfrak{g}}\xi)^{k+1} \neq 0$. Consider the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Using $\mathbb{C}$-linear extension, we regard $\mathfrak{g}^* \subseteq \mathfrak{g}_{\mathbb{C}}^*$. As such, our specific ξ still satisfies the condition $(\mathrm{d}_{\mathfrak{g}_{\mathbb{C}}}\xi)^{k+1} \neq 0$. Thus, the set

$$U := \{\eta \in \mathfrak{g}_{\mathbb{C}}^* : (\mathrm{d}_{\mathfrak{g}_{\mathbb{C}}}\eta)^{k+1} \neq 0\},$$

is non-empty, hence open and dense (because it is Zarisky open).

Claim: *For all $\eta \in \mathfrak{g}_{\mathbb{C}}^*$, we still have that*

$$\eta \wedge (\mathrm{d}_{\mathfrak{g}_{\mathbb{C}}} \eta)^{k+1} = 0. \tag{$*$}$$

Since extending forms $\mathbb{C}$-multilinearly yields a map of differential graded commutative algebras

$$(\wedge^\bullet \mathfrak{g}^*, \wedge, \mathrm{d}_{\mathfrak{g}}) \to (\wedge^\bullet_{\mathbb{C}} \mathfrak{g}^*_{\mathbb{C}}, \wedge, \mathrm{d}_{\mathfrak{g}_{\mathbb{C}}}),$$

Equation $(*)$ holds for elements of $\mathfrak{g}^*$. Thus, it is enough to show that, if $\nu, \mu \in \mathfrak{g}^*$, then also $\nu + i\mu$ satisfies $(*)$. Consider the map

$$\begin{aligned} q\colon \mathbb{C} &\to \wedge^{2k+3}_{\mathbb{C}} \mathfrak{g}^*_{\mathbb{C}} \\ z &\mapsto (\nu + z\mu) \wedge (\mathrm{d}_{\mathfrak{g}_{\mathbb{C}}}(\nu + z\mu))^{k+1}. \end{aligned}$$

Then q is a complex polynomial that vanishes for all $z \in \mathbb{R}$. Therefore, q vanishes identically, proving the claim.

Lemma 4.7.12 clearly also holds for complex Lie algebras. By $(*)$, any element $\eta \in U$ is of type (2). Since this is equivalent to (2") in Lemma 4.7.12, there exists $Y \in \mathfrak{g}_{\mathbb{C}}$ such that

$$\eta = \mathrm{ad}^*_Y \eta. \tag{4.19}$$

Let $B\colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ denote the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Then, for $\eta = B^\sharp X$, (4.19) is equivalent to

$$X = [X, Y].$$

Indeed, this follows because $B^\sharp$ is equivariant, and so

$$\mathrm{ad}^*_Y B^\sharp X = -B^\sharp(\mathrm{ad}_Y(X)) = B^\sharp([X, Y]).$$

On the other hand, by [Kna02, Section II, 2] the set of regular elements in $\mathfrak{g}_{\mathbb{C}}$ is given by the non-vanishing of a polynomial, hence it is also open and dense. Therefore, there exists a regular $X \in \mathfrak{g}_{\mathbb{C}}$ such that $B^\sharp X \in U$. Since X is regular, by [Kna02, Section II, Theorem 2.9],

$$\mathfrak{h} := \{H \in \mathfrak{g}_{\mathbb{C}} : [X, H] = 0\}$$

is a Cartan subalgebra. Consider the corresponding root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \oplus \mathfrak{h}.$$

Since $B^\sharp X \in U$, there exists

$$Y = \sum_{\alpha \in \Phi} Y_\alpha + Y_0 \in \mathfrak{g}_{\mathbb{C}},$$

such that $X = [X, Y]$. Since $X \in \mathfrak{h}$, we obtain a contradiction:

$$X = [X, Y] = \sum_{\alpha \in \Phi} \alpha(X) Y_\alpha \in \mathfrak{h} \cap \big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \big) = \{0\},$$

which concludes the proof. □

Lemma 4.8.5 *A semisimple Lie algebra of constant height is compact.*

Proof. Let $\mathfrak{g}$ be a semisimple Lie algebra of constant height. Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be a Cartan involution with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}, \qquad \mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}.$$

If B denotes the Killing form, then the two-form

$$B_\theta(X,Y) := -B(X,\theta(Y)), \qquad X,Y \in \mathfrak{g},$$

is positive definite and symmetric, and for any $Y \in \mathfrak{p}$ the operator $\mathrm{ad}_Y : \mathfrak{g} \to \mathfrak{g}$ is self-adjoint, see [Kna02, Lemma 6.27]. Thus, ad_Y is diagonalisable over $\mathbb{R}$. Let $X \in \mathfrak{g}$ be an eigenvector of ad_Y with eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$. Then

$$[X, -\tfrac{1}{\lambda}Y] = X.$$

As we have seen in the proof of Lemma 4.8.4, this is equivalent to $B^\sharp(X)$ being of type (2). This contradicts Lemma 4.8.4. Hence, all eigenvalues of ad_Y are zero. Since ad_Y is diagonalizable, Y must lie in the centre of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $Y = 0$. We have shown that $\mathfrak{p} = \{0\}$, which by [DK00, Theorem 3.6.2] implies that $\mathfrak{g}$ is compact, as its Killing form is negative definite. □

Theorem 4.8.6 *Any semisimple Lie algebra of constant height is isomorphic to $\mathfrak{so}(3)$.*

Proof. Let $\mathfrak{g}$ be semisimple of constant height k. By Lemma 4.8.4, all elements of $\mathfrak{g}^*$ have type (1). Hence, by Lemma 4.7.12, all nontrivial coadjoint orbits of $\mathfrak{g}^*$ have dimension $2k$. The Killing form induces an isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ of $\mathfrak{g}$-representations. So in the notation of [DK00, Definition (2.8.3)],

$$\mathfrak{g}^{\mathrm{reg}} = \mathfrak{g} \setminus \{0\}. \tag{$*$}$$

Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra. Since by Lemma 4.8.5 $\mathfrak{g}$ is compact, the complexification $\mathfrak{t}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, see [Kna02, Proposition 6.47], and corresponding roots $\alpha \in \Phi$ take real values on $i\mathfrak{t}$, see [Kna02, Corollary 4.49]. By [DK00, Theorem (3.7.1) (ii)] we have

$$\mathfrak{t} \setminus \{0\} \overset{(*)}{=} \mathfrak{g}^{\mathrm{reg}} \cap \mathfrak{t} = \mathfrak{t} \setminus \bigcup_{\alpha \in \Phi} \ker \alpha.$$

Since $\alpha\colon i\mathfrak{t} \to \mathbb{R}$, this implies $\dim \mathfrak{t} = 1$, i.e. $\mathfrak{g}_\mathbb{C}$ has rank 1 and is thus isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Its compact real form is $\mathfrak{so}(3)$, which proves the statement. □

With Theorem 4.8.6 at hand, we can proceed with the proof of Theorem 4.8.1. First, we show that constant height Lie algebras have to be abelian extensions of $\mathbb{R}$ or $\mathfrak{so}(3)$.

Lemma 4.8.7 *Let $\mathfrak{g}$ be a Lie algebra of constant height k and $\mathfrak{h} \subsetneq \mathfrak{g}$ a proper ideal. Then:*

1. *The height of $\mathfrak{g}/\mathfrak{h}$ is constant, equal to that of $\mathfrak{g}$.*

2. The ideal $\mathfrak{h}$ is abelian.

Proof. The first part follows from the fact that the identification

$$(\mathfrak{g}/\mathfrak{h})^* = \mathfrak{h}^{\mathrm{ann}} \subseteq \mathfrak{g}^*$$

induces an inclusion of differential graded commutative algebras

$$\big(\wedge^\bullet (\mathfrak{g}/\mathfrak{h})^*, \wedge, \mathrm{d}_{\mathfrak{g}/\mathfrak{h}}\big) \hookrightarrow \big(\wedge^\bullet \mathfrak{g}^*, \wedge, \mathrm{d}_{\mathfrak{g}}\big).$$

For the second part, choose a complement $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{c}$. For all $\theta \in \mathfrak{h}^{\mathrm{ann}} \setminus \{0\}$ and $\xi \in \mathfrak{h}^* = \mathfrak{c}^{\mathrm{ann}}$ we have

$$(\theta + \xi) \wedge (\mathrm{d}_{\mathfrak{g}}(\theta + \xi))^{k+1} = 0.$$

For a fixed θ, this equation is a polynomial in (the components of) ξ. Thus, to vanish, each homogeneous component needs to vanish separately. The linear terms read

$$\xi \wedge (\mathrm{d}_{\mathfrak{g}}\theta)^{k+1} + (k+1)\theta \wedge (\mathrm{d}_{\mathfrak{g}}\theta)^k \wedge \mathrm{d}_{\mathfrak{g}}\xi.$$

Using that $\mathrm{d}_{\mathfrak{g}}\theta \in \wedge^2\mathfrak{h}^{\mathrm{ann}}$, applying $\mathrm{i}_X\mathrm{i}_Y$ to this equation, where $X, Y \in \mathfrak{h}$, gives

$$(k+1)\xi([X,Y])\theta \wedge (\mathrm{d}_{\mathfrak{g}}\theta)^k = 0.$$

Since this holds for all $\xi \in \mathfrak{h}^*$, the assumption $\theta \wedge (\mathrm{d}_{\mathfrak{g}}\theta)^k \neq 0$ implies $[X,Y] = 0$. ☐

Corollary 4.8.8 *A Lie algebra of constant height has height either* 0 *or* 1.

Proof. Let $\mathfrak{h} \subsetneq \mathfrak{g}$ be a maximal proper ideal (which is necessarily abelian by Lemma 4.8.7). Then $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra of constant height with no proper ideals, so either $\mathfrak{g}/\mathfrak{h}$ is simple or $\mathfrak{g}/\mathfrak{h} = \mathbb{R}$. In the latter case the height of $\mathfrak{g}$ clearly is 0. If $\mathfrak{g}/\mathfrak{h}$ is simple, then $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{so}(3)$ by Theorem 4.8.6. ☐

Lemma 4.8.9 *A Lie algebra of constant height* 0 *is either abelian or is isomorphic to* $\mathbb{R} \ltimes \mathbb{R}^n$*, for the diagonal representation of* $\mathbb{R}$ *on* $\mathbb{R}^n$*.*

Proof. Let $\mathfrak{h} \subsetneq \mathfrak{g}$ be a maximal proper ideal. By Lemma 4.8.7, $\mathfrak{h}$ is abelian and $\mathfrak{g}/\mathfrak{h}$ is one-dimensional. Let $e \in \mathfrak{g}/\mathfrak{h} \setminus \{0\}$ and $X \in \mathfrak{g}$ be a preimage of e under the quotient map. Then we can identify $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{h}$, and the Lie bracket on $\mathfrak{g}$ is completely determined by the linear map

$$A := [X, \cdot\,] \colon \mathfrak{h} \to \mathfrak{h}.$$

Let $\theta \in (\mathbb{R}X)^\circ = \mathfrak{h}^*$ be given. By the assumption on $\mathfrak{g}$, we have $\theta \wedge \mathrm{d}_{\mathfrak{g}}\theta = 0$. Inserting X yields

$$0 = \mathrm{i}_X(\theta \wedge \mathrm{d}_{\mathfrak{g}}\theta) = -\theta \wedge \mathrm{i}_X\mathrm{d}_{\mathfrak{g}}\theta = \theta \wedge A^*\theta.$$

Hence, $A^*\theta = \lambda_\theta\theta$, for some $\lambda_\theta \in \mathbb{R}$. Therefore, any vector of $\mathfrak{h}^*$ is an eigenvector of A^*. Thus, the eigenvalues have to coincide, i.e. there exists $\lambda \in \mathbb{R}$, such that

$$A^* = \lambda \mathrm{id}_{\mathfrak{h}^*} \Leftrightarrow A = \lambda \mathrm{id}_{\mathfrak{h}}.$$

If $\lambda = 0$, then $\mathfrak{g}$ is abelian. If $\lambda \neq 0$, by rescaling X, we obtain the isomorphism $\mathfrak{g} \simeq \mathbb{R} \ltimes \mathfrak{h}$, for the diagonal representation of $\mathbb{R}$ on $\mathfrak{h}$. ☐

Finally, we show that nontrivial abelian extensions of $\mathfrak{so}(3)$ do not have constant height.

Lemma 4.8.10 *Any Lie algebra of constant height* 1 *is isomorphic to* $\mathfrak{so}(3)$.

Proof. Let $\mathfrak{g}$ be a Lie algebra of constant height 1. By the proof of Corollary 4.8.8 there exists a short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{so}(3) \to 0, \tag{4.20}$$

where $\mathfrak{h}$ is abelian. The adjoint representation of $\mathfrak{g}$ restricted to $\mathfrak{h}$ descends to a representation $\rho\colon \mathfrak{so}(3) \to \mathfrak{gl}(\mathfrak{h})$.

The obstruction to find a splitting $\sigma\colon \mathfrak{so}(3) \to \mathfrak{g}$ of (4.20) compatible with Lie brackets lies in the second cohomology group of $\mathfrak{so}(3)$ with coefficients in $\mathfrak{h}$. By the Whitehead Lemma, $\mathrm{H}^2(\mathfrak{so}(3), \mathfrak{h}) = 0$. Thus, we may assume that $\mathfrak{g}$ is a semidirect product

$$\mathfrak{g} = \mathfrak{so}(3) \ltimes \mathfrak{h}.$$

We show that the representation ρ of $\mathfrak{so}(3)$ on $\mathfrak{h}$ is trivial.

Claim: *For all* $\xi \in \mathfrak{h}^*$ *we have*

$$\dim \operatorname{span}\{\xi, \rho_X^*\xi : X \in \mathfrak{so}(3)\} \leq 2.$$

Fix $\theta \in \mathfrak{so}(3)^*$. As in the proof of Lemma 4.8.7, the expression

$$(\theta + \xi) \wedge (\mathrm{d}_{\mathfrak{g}}(\theta + \xi))^2 = 0$$

is a polynomial in $\xi \in \mathfrak{h}^*$. Vanishing of the degree two component reads

$$\theta \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^2 + 2\xi \wedge \mathrm{d}_{\mathfrak{g}}\theta \wedge \mathrm{d}_{\mathfrak{g}}\xi = 0. \tag{$*$}$$

We evaluate this expression on elements of $\mathfrak{so}(3)$. Denote by $\{X_1, X_2, X_3\}$ the standard basis of $\mathfrak{so}(3)$ and by $\{\theta_1, \theta_2, \theta_3\}$ the dual basis. Then

$$\mathrm{i}_{X_i}\mathrm{d}_{\mathfrak{g}}\theta_j = -\theta_j([X_i, \cdot\,]) = \sum_{k=1}^{3} \varepsilon_{ijk}\theta_k$$

and, for $\xi \in \mathfrak{h}^*$,

$$\mathrm{i}_X\mathrm{d}_{\mathfrak{g}}\xi = -\xi([X, \cdot\,]) = -\xi(\rho_X(\,\cdot\,)) = -\rho_X^*\xi.$$

We take $(*)$ for $\theta = \theta_1$ and evaluate on X_1, X_2, and X_3:

$$\begin{aligned}
0 &= \mathrm{i}_{X_3}\mathrm{i}_{X_2}\mathrm{i}_{X_1}\big(\theta_1 \wedge (\mathrm{d}_{\mathfrak{g}}\xi)^2 + 2\xi \wedge \mathrm{d}_{\mathfrak{g}}\theta_1 \wedge \mathrm{d}_{\mathfrak{g}}\xi\big)\\
&= \mathrm{i}_{X_3}\mathrm{i}_{X_2}\big((\mathrm{d}_{\mathfrak{g}}\xi)^2 + 2\theta_1 \wedge \rho_{X_1}^*\xi \wedge \mathrm{d}_{\mathfrak{g}}\xi + 2\xi \wedge \mathrm{d}_{\mathfrak{g}}\theta_1 \wedge \rho_{X_1}^*\xi\big)\\
&= \mathrm{i}_{X_3}(-2\rho_{X_2}^*\xi \wedge \mathrm{d}_{\mathfrak{g}}\xi + 2\theta_1 \wedge \rho_{X_1}^*\xi \wedge \rho_{X_2}^*\xi + 2\xi \wedge \theta_3 \wedge \rho_{X_1}^*\xi)\\
&= -2\rho_{X_2}^*\xi \wedge \rho_{X_3}^*\xi - 2\xi \wedge \rho_{X_1}^*\xi.
\end{aligned}$$

In conclusion, for all $\xi \in \mathfrak{h}^*$,

$$\xi \wedge \rho_{X_1}^*\xi = \rho_{X_3}^*\xi \wedge \rho_{X_2}^*\xi.$$

Hence, for any three elements η_1, η_2, η_3 of the set

$$\{\xi, \rho^*_{X_1}\xi, \rho^*_{X_2}\xi, \rho^*_{X_3}\xi\}$$

we have $\eta_1 \wedge \eta_2 \wedge \eta_3 = 0$, proving the claim.

With this, we show that the representation ρ is actually trivial. Consider the integration of ρ to an action of the compact group $SU(2)$ on $\mathfrak{h}^*$. By compactness of $SU(2)$ there exists an invariant inner product on $\mathfrak{h}^*$ with induced norm $\|\cdot\|$. Let $\xi \in \mathfrak{h}^*$ and $r := \|\xi\|$. Let $\mathcal{O}_\xi$ denote the orbit of ξ. Then, since $\mathcal{O}_\xi \subseteq S_r(0)$ is contained in a sphere,

$$\mathbb{R}\xi \cap T_\xi\mathcal{O}_\xi = \{0\}.$$

Using

$$T_\xi\mathcal{O}_\xi = \{\rho^*_X\xi : X \in \mathfrak{so}(3)\},$$

the Claim implies that $\dim T_\xi\mathcal{O}_\xi \leq 1$. Suppose that $\dim T_\xi\mathcal{O}_\xi = 1$. Then the isotropy Lie algebra of ξ is a two dimensional subalgebra of $\mathfrak{so}(3)$. But there exists no such subalgebra. Hence, $\dim T_\xi\mathcal{O}_\xi = 0$ and the representation has to be trivial.

In conclusion, $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{h}$ is a direct sum of Lie algebras. If $\mathfrak{h} \neq 0$, then $\mathfrak{so}(3) \subsetneq \mathfrak{g}$ is a proper ideal. By Lemma 4.8.7 $\mathfrak{so}(3)$ is abelian, a contradiction. Hence, $\mathfrak{h} = 0$ and $\mathfrak{g} \simeq \mathfrak{so}(3)$. ☐

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Chapter 5

On the real projective blowup of Poisson structures

This chapter contains [Sch25], accepted for publication in Differential Geometry and its Applications.

Abstract

We give a proof in the context of smooth differential geometry of Polishuk's theorem from 1997, in which he established under which conditions, given a Poisson scheme M and a Poisson subscheme N, the Poisson structure lifts to the blowup of M along N.

5.1 Introduction

Given a manifold M and a closed and embedded submanifold N, the **real projective blowup** is obtained by replacing N by $\mathbb{P}(\nu_N(M))$, the projectivisation of the normal bundle $\nu_N(M) = TM\big|_N/TN \to N$ of N in M. This yields a smooth manifold (without boundary), which we denote by $\mathrm{Blup}(M, N)$. The blowup comes with a canonical blowdown map

$$p\colon \mathrm{Blup}(M, N) \to M,$$

which on $\mathbb{P}(\nu_N(M))$ is the natural projection to N, and which is a diffeomorphism from the complement of $\mathbb{P}(\nu_N(M))$ to the complement of N. In particular, p is in general not a submersion. If $\operatorname{codim} N = 1$, then $p\colon \mathrm{Blup}(M, N) \to M$ is a diffeomorphism.

One can address the question of when geometric structures on M "lift" to the blowup. In [Pol97], Polishchuk studied the blowup of Poisson schemes, i.e. the existence of a Poisson structure on the blowup such that the blowdown map is Poisson. We state the result of [Pol97, Section 8] in its weaker, differential-geometric formulation, in a similar way to [GL13, Section 2.2].

Recall that, given a Poisson submanifold N of a Poisson manifold, at every point $y \in N$ the conormal space $(T_yN)^{\mathrm{ann}}$ carries a Lie bracket.

Theorem 5.1.1 (Polishchuk [Pol97]) *Let (M,π) be a Poisson manifold and $N \subseteq M$ a closed and embedded invariant (i.e. Poisson) submanifold. There exists a Poisson bivector field $\tilde{\pi}$ on* $\mathrm{Blup}(M,N)$ *which is p-related to π if and only if each Lie algebra $(T_yN)^{\mathrm{ann}}$, for $y \in N$, is either abelian or isomorphic to the semidirect product by the diagonal representation $\lambda \mapsto \lambda \mathrm{id}_{\mathbb{R}^n}$ of $\mathbb{R}$ on $\mathbb{R}^n$, denoted by $\mathbb{R} \ltimes \mathbb{R}^n$.*

Moreover, if $N \subseteq M$ is a closed and embedded submanifold of codimension > 1 such that there exists a Poisson structure on $\mathrm{Blup}(M,N)$ *which is p-related to π, then N is necessarily invariant.*

The proof in [Pol97] is of algebraic nature. In this note, we provide an elementary proof of the statement in Section 5.4.

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5.2 Real projective blowups

Since we only need very selected background from Section 1.1.4, we list them again here.

The smooth structure of the real projective blowup can be described via the following set of charts. Let $(U,(x,y))$ be a submanifold chart for N, i.e. $U \cap N = \{x = 0\}$. Then the collection

$$U_k = p^{-1}\big(\{x_k \neq 0\}\big) \cup \big\{[v] \in \mathbb{P}(\nu_N(M)|_{U\cap N})\colon \mathrm{d}x_k(v) \neq 0\big\},$$

$k = 1,\dots,\operatorname{codim} N$, is an open cover of $p^{-1}(U)$. On each U_k one defines coordinates $(\tilde{x},y)$ on $\mathrm{Blup}(M,N)$, in which the blowdown map reads (see e.g. [Obs21, Remark 5.29] for details)

$$p(\tilde{x}_1,\dots,\tilde{x}_k,\dots,\tilde{x}_{\operatorname{codim} N},y) = (\tilde{x}_k\tilde{x}_1,\dots,\tilde{x}_k,\dots,\tilde{x}_k\tilde{x}_{\operatorname{codim} N},y). \tag{5.1}$$

Notice that the hyperplane $\mathbb{P}(\nu_N(M)) \cap U_i$ is given by $\tilde{x}_k = 0$, and that the chart we obtain there by restriction is the well-known chart on the projective bundle induced by the fibrewise linear coordinates $(\mathrm{d}x, y)$ on the normal bundle $\nu_N(M)$. Note that the image of Tp at a point $[v] \in \mathbb{P}(\nu_N(M))$, where $v \in \nu_N(M) \setminus \{0\}$, is given by

$$\mathrm{im}\,(T_{[v]}p) = T_{p([v])}N \oplus \mathbb{R}v. \tag{5.2}$$

On $M \setminus N$, one can pull back vector fields to $\mathrm{Blup}(M,N) \setminus \mathbb{P}(\nu_N(M))$ via the blowdown map. Given a submanifold chart $(U,(x,y))$ as above, in the chart $(U_k, (\tilde{x}, y))$ we have

$$\begin{aligned} p^*\Big(\frac{\partial}{\partial x_j}\big|_{U\setminus N}\Big)\big|_{U_k} &= \frac{1}{\tilde{x}_k}\frac{\partial}{\partial \tilde{x}_j} \qquad \text{for } j \neq k, \\ p^*\Big(\frac{\partial}{\partial x_k}\big|_{U\setminus N}\Big)\big|_{U_k} &= \frac{\partial}{\partial \tilde{x}_k} - \sum_{j\neq k} \frac{\tilde{x}_j}{\tilde{x}_k}\frac{\partial}{\partial \tilde{x}_j}, \\ p^*\Big(\frac{\partial}{\partial y}\Big)\big|_{U_i} &= \frac{\partial}{\partial y} \end{aligned} \tag{5.3}$$

as one sees by applying these vector fields on functions of the form $p^* x_j$. This implies the following standard result (see Lemma 4.2.1).

Lemma 5.2.1 *Given a vector field X on M, there exists a vector field $\tilde{X}$ on* $\mathrm{Blup}(M, N)$ *which is p-related to X if and only if X is tangent to N. In that case, $\tilde{X}$ is unique and tangent to $\mathbb{P}(\nu_N(M)) \subseteq \mathrm{Blup}(M, N)$.*

5.3 Linearisation of Poisson structures

Given a bivector field $\pi \in \Gamma(\wedge^2 TM)$ and an invariant submanifold $N \subseteq M$, i.e. $\pi\big|_N \in \Gamma(\wedge^2 TN)$, the linearisation $\pi_{\mathrm{lin}} \in \Gamma(\wedge^2 T\nu_N(M))$ is defined as the linear bivector field on the vector bundle $\mathrm{pr}\colon \nu_N(M) \to N$, such that

$$\begin{aligned} \pi_{\mathrm{lin}}(\mathrm{pr}^*\phi, \cdot\,) &= 0 \qquad \text{and} \\ \pi_{\mathrm{lin}}(\mathrm{d}f\big|_N, \mathrm{d}g\big|_N) &= \mathrm{d}(\pi(\mathrm{d}f, \mathrm{d}g))\big|_N \end{aligned}$$

for all $\phi \in \mathscr{C}^\infty(N)$, and $f, g \in \mathscr{C}^\infty(M)$ that vanish on N. Let $(U, (x, y))$ be a submanifold chart for N with local form of π given by

$$\pi\big|_U = \frac{1}{2}\sum_{i,j} \pi_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{i,\alpha} \pi_{i\alpha}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_\alpha} + \frac{1}{2}\sum_{\alpha,\beta} \pi_{\alpha\beta}\frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta},$$

where the coefficients are smooth functions of (x, y), with $\pi_{ij} = -\pi_{ji}$ and $\pi_{\alpha\beta} = -\pi_{\beta\alpha}$. Invariance of N implies $\pi_{ij}(0, y) = 0 = \pi_{i\alpha}(0, y)$. After possibly shrinking we can identify $\nu_{N\cap U}(U) = U$ and then the Poisson structure π_{lin} becomes the linearisation of the first term

$$\pi_{\mathrm{lin}} = \frac{1}{2}\sum_{i,j,\ell} x_\ell \frac{\partial \pi_{ij}}{\partial x_\ell}(0, y)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{5.4}$$

which implies the following.

Lemma 5.3.1 *Identifying $\nu_N(M)$ with a neighbourhood of N, the bivector field $\pi - \pi_{\mathrm{lin}}$ is given by the sum of $\wedge$-products of vector fields that are tangent to N.*

If N is an invariant submanifold for a Poisson structure π, recall that there is a Lie algebra structure on the conormal space

$$(T_y N)^{\mathrm{ann}} = (T_y M / T_y N)^* \tag{5.5}$$

given by

$$[(\mathrm{d}f)\big|_N(y), (\mathrm{d}g)\big|_N(y)] := (\mathrm{d}\{f,g\})\big|_N(y), \tag{5.6}$$

where f, g are smooth functions on M vanishing on N.

In the case that $N \subseteq M$ is a leaf of the Poisson structure, all conormal Lie algebra structures are isomorphic.

On the other hand, π_{lin} is a linear Poisson structure on $\nu_N(M)$, turning $(TN)^{\text{ann}} = \nu_N(M)^*$ into a bundle of Lie algebras.

Lemma 5.3.2 *The Lie algebra structure induced by* π_{lin} *coincides with* (5.6).

Proof. Identifying $\nu_N(M)$ with a neighbourhood of N, by Lemma 5.3.1, for any $f, g \in \mathscr{C}^\infty(M)$ vanishing on N the function

$$(\pi - \pi_{\text{lin}})(\mathrm{d}f, \mathrm{d}g)$$

vanishes quadratically along N. □

5.4 Lifting of Poisson structures

In [Pol97, Section 8], Polishchuk used algebraic methods to prove a result on the blowup of Poisson structures on Poisson schemes, which was translated to smooth manifolds in [GL13, Section 2.2]. We give a direct proof of the differential-geometric version [GL13] of Polishchuk's result using coordinates on the blowup.

Proposition 5.4.1 *Let* (M, π) *be a Poisson manifold and* $N \subseteq M$ *a closed and embedded invariant submanifold, i.e.* $\pi\big|_N \in \Gamma(\wedge^2 TN)$. *Then there exists a Poisson bivector field* $\tilde{\pi}$ *on* $\operatorname{Blup}(M, N)$ *p-related to* π *iff the linearisation of* π *along* N *is given by*

$$\pi_{\text{lin}} = \xi \wedge v^{\text{ver}}, \tag{5.7}$$

where by ξ *we denote the Euler vector field on the normal bundle* $\nu_N(M) = TM\big|_N / TN$ *and* v^{ver} *is the vertical lift of a section* $v \in \Gamma(\nu_N(M))$, *i.e. the constant extension of* v *to* $\nu_N(M)$.

Moreover, $\mathbb{P}(\nu_N(M)) = p^{-1}(N)$ *is a Poisson submanifold iff* $\pi_{\text{lin}} = 0$.

Remark 5.4.2 Starting from a Poisson structure π on M, once there exists a bivector field $\tilde{\pi}$ on $\operatorname{Blup}(M, N)$ which is p-related to π, it is automatically Poisson, as the (closed) integrability condition $[\![\tilde{\pi}, \tilde{\pi}]\!]_{\mathrm{S}} = 0$ holds on an open and dense subset, and it is necessarily unique, as it is fixed on a dense set. On the other hand, the Poisson condition never enters in the proof, hence Proposition 5.4.1 and Lemma 5.4.3 below hold for any bivector field π on M.

Moreover, in addition to Proposition 5.4.1 one can show that invariance of N is a necessary condition for a lift of π to exist, provided that $\operatorname{codim} N > 1$, see [Pol97, Proposition 8.3].

Lemma 5.4.3 *Let* $\pi \in \Gamma(\wedge^2 TM)$ *and* $N \subseteq M$ *be a closed and embedded submanifold with* $\operatorname{codim} N > 1$, *such that there exists a bivector field on* $\operatorname{Blup}(M, N)$ *p-related to* π. *Then* N *is invariant.*

Proof. Let $\tilde{\pi}$ be a bivector field on $\mathrm{Blup}(M, N)$ which is p-related to π. Then, for all $y \in N$ we have

$$\bigcap_{\xi \in p^{-1}(y)} \operatorname{im}(T_\xi p) = T_y N$$

as $\operatorname{codim} N > 1$, so $\pi\big|_N \in \Gamma(\wedge^2 TN)$ follows. ☐

Proof of Proposition 5.4.1. If $\operatorname{codim} N = 1$ there is nothing to show. Note that the problem is local. By Lemma 5.2.1 and 5.3.1 we can assume

$$\pi = \pi_{\mathrm{lin}} = \frac{1}{2} \sum_{i,j,\ell} x_\ell \frac{\partial \pi_{ij}}{\partial x_\ell}(0, y) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{5.8}$$

Moreover, we write

$$\pi_\ell^{ij} := \frac{\partial \pi_{ij}}{\partial x_\ell}(0, \,\cdot\,)$$

as a function of y. In the chart $(U_k, \tilde{x})$ the coefficient functions of $\tilde{\pi} = p^*\pi\big|_{U_k \setminus \mathbb{P}(\nu_N(M))}$ are given by

$$\frac{1}{2}\tilde{\pi}_{ij} = \frac{1}{2\tilde{x}_1}\left[\pi^1_{ij} - \tilde{x}_j \pi^1_{i1} - \tilde{x}_i \pi^1_{1j} + \sum_{k \neq 1} \tilde{x}_k \left(\pi^k_{ij} - \tilde{x}_j \pi^k_{i1} - \tilde{x}_i \pi^k_{1j}\right)\right] \text{ if } i, j \neq 1,$$

and

$$\frac{1}{2}\tilde{\pi}_{ij} = \frac{1}{2}\Big(\pi^1_{ij} + \sum_{k \neq 1} \tilde{x}_k \pi^k_{ij}\Big) \text{ if } i = 1 \text{ or } j = 1,$$

as follows from (5.3). These functions smoothly extend to $U_k \cap \mathbb{P}(\nu_N(M))$ (hence define a Poisson bivector field on U_k) iff the polynomials on $U_k \cap \mathbb{P}(\nu_N(M))$ given by

$$\pi_k^{ij} - \tilde{x}_j \pi_k^{ik} - \tilde{x}_i \pi_k^{kj} + \sum_{\ell \neq k} \tilde{x}_\ell \left(\pi_\ell^{ij} - \tilde{x}_j \pi_\ell^{ik} - \tilde{x}_i \pi_\ell^{kj}\right),$$

where $i, j \neq k$, vanish identically for all $y \in N$ (these are just the expressions in the square bracket above). Considering the constant term we find

$$\pi_k^{ij} = 0.$$

From the vanishing of the quadratic term, if in the sum $\ell \neq i$ and $\ell \neq j$ we find

$$\tilde{x}_\ell (\tilde{x}_j \pi_\ell^{ik} + \tilde{x}_i \pi_\ell^{kj}) = 0,$$

hence in this case $\pi_\ell^{ik} = 0 = \pi_\ell^{kj}$. The remainder of the quadratic terms read

$$\tilde{x}_i^2 \pi_i^{kj} + \tilde{x}_j^2 \pi_j^{ik} + \tilde{x}_i \tilde{x}_j (\pi_i^{ik} + \pi_j^{kj}) = 0,$$

thus, $\pi_i^{kj} = 0 = \pi_j^{ik}$ and $\pi_i^{ik} = c^k$ is a function depending on k and smoothly on y. Finally, from the linear terms we obtain $\pi_\ell^{ij} = 0$. Summarising, the functions $\tilde{\pi}^{ij}$ define a Poisson bivector field on U_k iff

$$\pi_j^{ik} = \delta_j^i c^k.$$

This is equivalent to the linearisation of π to be

$$\pi_{\mathrm{lin}} = \frac{1}{2}\sum_{i,j,\ell} x_\ell \pi_\ell^{ij} \partial_i \wedge \partial_j = \sum_{i,j} x_i \partial_i \wedge c^j \partial_j = \xi \wedge v^{\mathrm{ver}}, \tag{5.9}$$

where $v^{\mathrm{ver}} = \sum_j c^j \partial_j$.

Now, suppose that π is of this type. Then on $U_k \cap \mathbb{P}(\nu_N(M))$,

$$\tilde{\pi}^{ki} = c^i,$$

hence $\mathbb{P}(\nu_N(M))$ is a Poisson submanifold iff $v = 0$. □

Next, we interpret the condition $\pi_{\mathrm{lin}} = \xi \wedge v^{\mathrm{ver}}$ from Proposition 5.4.1 algebraically to recover the differential-geometric version of [Pol97, Proposition 8.1].

Proposition 5.4.4 *Let (M, π) be a Poisson manifold and $N \subseteq M$ a closed and embedded invariant submanifold. Then there exists a Poisson structure on* $\mathrm{Blup}(M, N)$ *p-related to π if and only if every fibre of the bundle of Lie algebras $(TN)^{\mathrm{ann}} \to N$ is isomorphic to either of the following.*

1. *An abelian Lie algebra,*
2. *The semidirect product $\mathbb{R} \ltimes V$, where the representation of $\mathbb{R}$ on the vector space V is the diagonal representation.*

Proof. Assume that π lifts to the blowup. By Lemma 5.3.2 and Proposition 5.4.1 we can assume

$$\pi = \pi_{\mathrm{lin}} = \xi \wedge v^{\mathrm{ver}},$$

where ξ denotes the Euler vector field of $\nu_N(M)$, $v \in \Gamma(\nu_N(M))$, and $\cdot^{\mathrm{ver}}$ is the vertical lift. Let $y \in N$.

If $v_y = 0$, then clearly the induced Lie algebra structure is abelian. If $v_y \neq 0$, let $(x_1, \ldots, x_n, y)$ denote fibrewise linear coordinates on $\nu_N(M)$, where $\{x = 0\}$ is the zero section and $x_i(v) = \delta_{i1}$.

Then

$$\pi_{\mathrm{lin}} = \sum_{i=2}^{n} x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_1},$$

and, by inserting in (5.6) it is immediate to see that

$$\begin{aligned} [\mathrm{d}x_i, \mathrm{d}x_j] &= 0 \quad \text{if } i, j \neq 1, \text{ and} \\ [\mathrm{d}x_i, \mathrm{d}x_1] &= \mathrm{d}x_i, \end{aligned}$$

which is isomorphic to the Lie algebra of the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^{n-1}$, where the representation of $\mathbb{R} = \mathbb{R}\mathrm{d}x_1$ is given by $\mathrm{d}x_1 \mapsto \mathrm{id}_{\mathbb{R}^{n-1}}$. Conversely, assume that each fibre of the bundle $(TN)^{\mathrm{ann}} \to N$ is either isomorphic to an abelian Lie algebra or $\mathbb{R} \ltimes V$. Going the same computation backwards, we obtain a set-theoretic section $v\colon N \to \nu_N(M)$, such that the bundle of Lie algebras corresponds to $\pi_{\mathrm{lin}} = \xi \wedge v^{\mathrm{ver}}$ (with the zeros of v corresponding to abelian bundles). Since π_{lin} is smooth and the components of v depend on the coefficient functions of π via (5.9), v is smooth. □

5.5 Interpretation of Equation (5.7)

Remark 5.5.1 Conceptually, one can make sense of the form (5.7) in the following way. If π is "quadratically tangent" to N, i.e. we can write it as (the sum of) the product of two vector fields that are tangent to N, we can lift those vector fields separately to lift π. However, if π is only "linearly tangent" to N, then we can only lift one of the two factors of π using Lemma 5.2.1. This lifted vector field must vanish along $\mathbb{P}(\nu_N(M))$ to make up for the singularity that the other factor gains on the blowup. Hence, by Lemma 5.5.2 below the linearisation of the tangent factor of π must be proportional to the Euler vector field.

Lemma 5.5.2 *Given a vector field X tangent to N, the restriction of the lift $\tilde{X}$ to $\mathbb{P}(\nu_N(M))$ is the projectivisation of X_{lin}, the fibrewise linear vector field on $\nu_N(M)$ obtained by linearising X along N. In particular, $\tilde{X}\big|_{\mathbb{P}(\nu_N(M))} = 0$ if and only if X_{lin} is a $\mathscr{C}^\infty(N)$-multiple of the Euler vector field.*

Proof. We make use of the *deformation to the normal cone* construction for $\operatorname{Blup}(M, N)$ ([DS21, Section 2.8]). As a set, the deformation to the normal cone is given by

$$\operatorname{DNC}(M, N) = M \times (\mathbb{R} \setminus \{0\}) \sqcup (\nu_N(M) \times \{0\})$$

and carries a smooth ($\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$)-action given by

$$\lambda.(v, t) = \begin{cases} (v, \lambda^{-1} t) & \text{if } t \neq 0, \\ (\lambda v, 0) & \text{if } t = 0. \end{cases} \tag{5.10}$$

The blowup of N in M is then given by the quotient

$$\operatorname{Blup}(M, N) = \frac{\operatorname{DNC}(M, N) \setminus (N \times \mathbb{R})}{\mathbb{R}^\times} = M \setminus N \cup \mathbb{P}(\nu_N(M)).$$

Moreover, the diagram

$$\begin{tikzcd} \operatorname{DNC}(M, N) \setminus (N \times \mathbb{R}) \arrow[r] \arrow[dr, "\kappa"] & \operatorname{Blup}(M, N) \arrow[d, "p"] \\ & M \end{tikzcd} \tag{5.11}$$

commutes, where κ is the restriction of $(m, t) \mapsto m$ on $M \times (\mathbb{R} \setminus \{0\})$ and of the vector bundle projection $\nu_N(M) \to N \subseteq M$ at $t = 0$.

Any vector field X on M tangent to N gives rise to a vector field $\mathcal{D}(X)$ on $\operatorname{DNC}(M, N)$, which on $M \times (\mathbb{R} \setminus \{0\})$ is $X \times 0$ and which on $\nu_N(M)$ is the linearisation of X at N [BBLM20, Section 2.2, Property e)]. The vector field $\mathcal{D}(X)$ is invariant under the $\mathbb{R}^\times$ action (5.10), since it is clearly invariant on $M \times (\mathbb{R} \setminus \{0\})$. Hence, $\mathcal{D}(X)$ descends to a vector field on $\operatorname{Blup}(M, N)$. This vector field is precisely $\tilde{X}$, because of the commutativity of (5.11), the fact that $\mathcal{D}(X)$ is κ-related to X, and that the blowdown map is a diffeomorphism on an open dense set. Restricting to points of $\mathbb{P}(\nu_N(M)) = (\nu_N(M) \setminus N)/\mathbb{R}^\times$, the conclusion follows. ☐

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Chapter 6

A note on pullbacks and blowups of Lie algebroids, singular foliations, and Dirac structures

This chapter contains [SZ25], accepted for publication in Differential Geometry and its Applications. My contributions lie mainly in Section 6.5.2.

Abstract

Lie algebroids, singular foliations, and Dirac structures are closely related objects. We examine the relation between their pullbacks under maps satisfying a transversality assumption. A special case is given by blowdown maps. In that case, we also establish the relation between the blowup of a Lie algebroid and its singular foliation.

6.1 Introduction

Dirac structures are a kind of geometric structure on manifolds that generalizes both closed 2-forms and Poisson structures. An aim of this note is to determine the singular foliation underlying the pullback of a Dirac structure D under a transverse map, by showing that it coincides with the pullback of the singular foliation of D. Here, "singular foliation" is understood as a submodule of vector fields, in the sense of [AS09], and not just as the underlying partition of the manifold into leaves. We do this in Corollary 6.4.4. The analogue statement for leaves is known and was given in [Măr16, Section 4].

To obtain Corollary 6.4.4, we involve Lie algebroids. It is well-known that any Lie algebroid A induces a singular foliation $\mathcal{F} := \sharp(\Gamma_c(A))$, where $\sharp$ denotes the anchor, and the subscript c compact support. Further, any Dirac structure inherits a Lie algebroid structure, with anchor the restriction of pr_{TM}, and bracket the restriction of the Courant bracket.

Let $f\colon B \to M$ be a smooth map transverse to a Dirac structure or Lie algebroid over M. In that case, one can consider the pullback Dirac structure, the pullback Lie algebroid, and the pullback singular foliation. In the first part of this note we compare these objects, recovering results that are certainly known to the experts but for which we could find only partial accounts and proofs in the literature (see "Relation with the literature" below). Namely, the following operations commute with taking pullbacks:

- taking the singular foliation of a Lie algebroid (see Proposition 6.3.1),
- taking the Lie algebroid underlying a Dirac structure (see Proposition 6.4.1).

Corollary 6.4.4 immediately follows from this.

In the second part of this note we consider a submanifold $N \subseteq M$ and the blowup $\mathrm{Blup}(M,N)$ of M along N. The blowdown map $p\colon \mathrm{Blup}(M,N) \to M$ is a diffeomorphism on an open dense subset. If a Dirac structure D on M is transverse to N, it admits a unique lift to $\mathrm{Blup}(M,N)$. Using the above results we show that the singular foliation of the lift is the pullback of the singular foliation of D.

If a Lie algebroid A on M is transverse to N, then it gives rise to several Lie algebroids on B: one of them is the pullback Lie algebroid $p^!A$, others are obtained blowing up A itself with respect to a Lie subalgebroid L supported on N. We compute the singular foliation $\mathcal{F}_{\mathrm{Blup}}$ of the blown-up Lie algebroid in two cases:

- when L is the restriction of A to N, $\mathcal{F}_{\mathrm{Blup}}$ consists of lifts of vector fields on M tangent to N (see Proposition 6.5.2),
- when L is the isotropy Lie algebroid of A over N, $\mathcal{F}_{\mathrm{Blup}}$ consists of lifts of vector fields on M vanishing on N (see Proposition 6.5.5),

and finally generalise to any Lie subalgebroid over N which contains the isotropies of A over N.

Relation with the literature

The statement of Proposition 6.4.1 appears in [BLM16, Section 5.1], but no further details are given there. A proof can be obtained from the one of Proposition 6.6 in [Mei17a, Section 6.2], yielding an isomorphism which is the inverse of the one we construct in our proof of Proposition 6.4.1.

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6.2 Definitions of pullbacks

Let $f\colon B \to M$ be a smooth map.

6.2.1 Pullbacks of Lie algebroids

A **Lie algebroid** is a vector bundle A over a manifold M, together with a bundle map $\sharp\colon A \to TM$ called anchor and a Lie bracket on the smooth sections $\Gamma(A)$, such that $[a, fa'] = \sharp(a)(f)a' + f[a,a']$ for all sections a, a' and functions f. Lie algebroids play an important role in differential geometry, generalizing both tangent bundles and Lie algebras.

Let A be a Lie algebroid over M. Assume f is transverse (For a more general treatment under an assumption weaker than transversality, see [BCdH16, Appendix A.2]) to A, in the sense that for all $x \in B$, we have

$$\sharp(A_{f(x)}) + T_x f(T_x B) = T_{f(x)} M. \tag{6.1}$$

We denote by $f^\sharp A \to B$ the pullback of A as a vector bundle. The **pullback of A as a Lie algebroid** ([HM90], see also [Mac87, Section 4]) is

$$f^! A := f^\sharp A \times_{TM} TB,$$

the fibre product over $\sharp \circ f^\sharp\colon A \to TM$ and $Tf\colon TB \to TM$, where $f^\sharp\colon f^\sharp A \to A$ is the canonical map.

First of all, $f^! A$ is a vector bundle over B, of rank equal to $\operatorname{rank}(A) + (\dim B - \dim M)$. Indeed, $f^! A$ is the preimage of the diagonal under the map $(\sharp, TM)\colon f^* A \oplus TB \to TM \times TM$, and the assumption that A is transverse to f implies that this map is transverse to the diagonal. Any section of $f^! A$ is of the form $(\sum h_i (a_i \circ f), X)$ for finitely many $h_i \in \mathscr{C}^\infty(B)$ and $a_i \in \Gamma(A)$, and for $X \in \Gamma(TB)$ such that $f_* X = \sum h_i (\sharp(a_i) \circ f)$. Here, we view Tf as a map $f_*\colon TB \to f^* TM$ covering id_B.

More importantly, $f^! A$ is a Lie algebroid, with the second projection as anchor. The bracket of two sections as above is determined by the Lie algebroid bracket of A and the Leibniz rule:

$$\Big[\Big(\sum_i h_i(a_i \circ f), X\Big), \Big(\sum_j h'_j(a'_j \circ f), X'\Big)\Big] = (\star, [X, X']),$$

where

$$\star = \sum_{i,j} h_i h'_j([a_i, a'_j] \circ f) + \sum_j X(h'_j)(a'_j \circ f) - \sum_i X'(h_i)(a_i \circ f). \tag{6.2}$$

Indeed, $f^! A$ is a Lie subalgebroid over $\operatorname{graph}(f) \subseteq M \times B$ of the product Lie algebroid $A \times TB$, under the identification $B \simeq \operatorname{graph}(f)$ [Mei17b, Section 7.4][Mei24].

Further, the first projection yields a Lie algebroid morphism $f^! A \to A$ [Mac87, Section 4.3]. This morphism is not surjective in general; for instance, for any smooth map $f\colon B \to M$, we have $f^! TM = TB$, and the morphism is given by the derivative $Tf\colon TB \to TM$.

6.2.2 Pullbacks of singular foliations

A **singular foliation** $\mathcal{F} \subseteq \Gamma_c(TM)$ is a $\mathscr{C}^\infty(M)$-submodule of the compactly supported vector fields, which is locally finitely generated and involutive with respect to the Lie bracket [AS09]. A singular foliation gives rise to a decomposition of M into immersed submanifolds of possibly varying dimension, called leaves.

Suppose we have a singular foliation $\mathcal{F} \subseteq \Gamma_c(TM)$ which is transverse to f, i.e. each leaf of the singular foliation is transverse to f. Then there exists a **pullback singular foliation** [AS09, Section 1.2.3] given by

$$f^{-1}\mathcal{F} := \{X \in \Gamma_c(TB) : f_* X = \sum h_i(Y_i \circ f) \text{ for } h_i \in \mathscr{C}^\infty_c(B), Y_i \in \mathcal{F}\},$$

where the sum is finite. Here, we use the subscript c to denote "compactly supported". The leaves of $f^{-1}\mathcal{F}$ are the connected components of the preimages of the leaves of $\mathcal{F}$.

6.2.3 Pullbacks of Dirac structures

Recall that a **Dirac structure** $D \subseteq TM \oplus T^*M$ on M is a maximally isotropic subbundle (with respect to the canonical symmetric pairing) that is involutive with respect to the Courant bracket

$$[(X, \xi), (X', \xi')] = ([X, X'], \mathscr{L}_X \xi' - \mathrm{i}_{X'} \mathrm{d}\xi). \tag{6.3}$$

Prototypical examples of Dirac structures are graphs for closed 2-forms and of Poisson bivector fields.

Consider

$$\mathfrak{B}_f D := \{(v, f^*\eta) \in TB \oplus T^*B : (Tf(v), \eta) \in D\},$$

a collection of maximally isotropic subspaces of $TB \oplus T^*B$. Assume that $\sharp(D)$ is transverse to f, i.e. Equation (6.1) is satisfied for $D = A$, $\sharp = \mathrm{pr}_{TM}$. Then $\mathfrak{B}_f D$ is a Dirac structure on B [Bur13, Proposition 5.6.][Măr16, Section 4]. The map f from $(B, \mathfrak{B}_f D)$ to (M, D) is then said to be a **backward Dirac map**.

6.3 Lie algebroids and singular foliations

Under the transversality assumption, the operations of taking the singular foliation of a Lie algebroid and of taking the pullback commute.

Proposition 6.3.1 *Let $f \colon B \to M$ be a smooth map. Let A be a Lie algebroid over M, transverse to f. Denote by $\mathcal{F}_A := \sharp(\Gamma_c(A))$ the singular foliation of A, and $\mathcal{F}_{f^! A} := \sharp_{f^! A}(\Gamma_c(f^! A))$ the singular foliation of the pullback Lie algebroid.*

Then

$$\mathcal{F}_{f^! A} = f^{-1}\mathcal{F}_A.$$

Proof. "$\subseteq$" Any compactly supported section of $f^!A$ is of the form $(\sum h_i(a_i \circ f), X)$ for finitely many $h_i \in \mathscr{C}^\infty_{\mathrm{c}}(B)$ and $a_i \in \Gamma(A)$, and $X \in \Gamma_{\mathrm{c}}(TB)$ such that $f_*X = \sum h_i(\sharp(a_i) \circ f)$. We may assume that the a_i are compactly supported, by multiplying them with a compactly supported function which equals 1 on $f(\mathrm{supp}(h_i))$. Since $\sharp(a_i) \in \mathcal{F}_A$, the conclusion follows.

"$\supseteq$" Let $X \in f^{-1}\mathcal{F}_A$, i.e. $f_*X = \sum h_i(Y_i \circ f)$, where $h_i \in \mathscr{C}^\infty_{\mathrm{c}}(B), Y_i \in \mathcal{F}_A$. We have $Y_i = \sharp(a_i)$ for certain $a_i \in \Gamma_{\mathrm{c}}(A)$. Then $(\sum h_i(a_i \circ f), X) \in \Gamma_{\mathrm{c}}(f^!A)$, since $\sum h_i(\sharp(a_i) \circ f) = \sum h_i(Y_i \circ f) = f_*X$. This is an element of $\Gamma_{\mathrm{c}}(f^!A)$, whose image under the anchor is X. □

6.4 Dirac structures and Lie algebroids

Let D be a Dirac structure over M, transverse to a smooth map $f\colon B \to M$. Recall that we denote by $f^\sharp D$ and $f^\sharp TM$ the respective pullback vector bundles.

Proposition 6.4.1 *Let D be a Dirac structure over M, transverse to f. Consider $\mathfrak{B}_f D$, viewed as a Lie algebroid, and the Lie algebroid pullback $f^!D$ of D viewed as a Lie algebroid. Then the Lie algebroids $\mathfrak{B}_f D$ and $f^!D$ are canonically isomorphic, by an isomorphism covering* id_B.

Proof. Notice first that at a point $x \in B$, any element of $(f^!D)_x$ is of the form (ℓ, v) where $\ell \in D_{f(x)}$, $v \in T_xB$ with $T_xf(v) = \mathrm{pr}_{TM}\ell$, hence $\ell = (T_xf(v), \eta)$ for some $\eta \in T^*_{f(x)}M$. Consider the vector bundle map

$$\psi\colon f^!D \to \mathfrak{B}_f D, \quad ((T_xf(v), \eta), v) \mapsto (v, (T_xf)^*\eta). \tag{6.4}$$

This vector bundle map clearly takes values in $\mathfrak{B}_f D$. We first argue that ψ is a vector bundle isomorphism. It is surjective, by the very definition of $\mathfrak{B}_f D$. We have $\mathrm{rank}\,(f^!D) = \dim(B)$ using $\mathrm{rank}\,(D) = \dim(M)$, see Section 6.2. Hence, ψ is a map between vector bundles of the same rank. Therefore, it is also injective, and an isomorphism.

The map ψ clearly preserves the anchors. To show that it preserves brackets, take a section

$$(\sum h_i((Y_i, \eta_i) \circ f), X) \in \Gamma(f^!D),$$

where $h_i \in \mathscr{C}^\infty(B)$ and $(Y_i, \eta_i) \in \Gamma(D)$, for $X \in \Gamma(TB)$ such that $f_*X = \sum h_i(Y_i \circ f)$. Under ψ, it is mapped to $(X, \sum h_i f^*\eta_i) \in \Gamma(\mathfrak{B}_f D)$. Use Equation (6.2) to compute the bracket of two sections of $\Gamma(f^!D)$, recalling that the Lie bracket of D is the restriction of the Courant bracket on M. Use the Courant bracket (6.3) on B to compute the bracket of their images under ψ, together with identities such as $\mathrm{i}_X f^*\eta = \sum_i h_i((\mathrm{i}_{Y_i}\eta) \circ f)$ and the fact that D is isotropic, to conclude that ψ preserves brackets. □

Remark 6.4.2 If in the proof of Proposition 6.4.1 we view ψ as a (injective) vector bundle map $f^!L \to TB \oplus T^*B$, we recover the facts that its image $\mathfrak{B}_f L$ is a smooth subbundle of $TB \oplus T^*B$ and that it is involutive.

Remark 6.4.3 We provide a direct proof that the map ψ in (6.4) is injective, without using any dimension considerations.

We first claim: **The vector bundle map over** id_B

$$\phi\colon f^\sharp D \to f^\sharp TM \oplus T^*B, \qquad (Y,\eta) \mapsto (Y,(f_*)^*\eta)$$

is injective.

Indeed, for all $x \in B$, the transversality condition (6.1) (for $A = D$) can be rephrased as $D_{f(x)} \cap [T_x f(T_x B)]^\circ = \{0\}$, as one sees taking annihilators and using $\sharp(D) = (D \cap TM)^\circ$. Now let $(v,\eta) \in (f^\sharp D)_x$ lie in the kernel of ϕ, i.e. $v = 0$ and $\eta \in [T_x f(T_x B)]^\circ$. Then $(v,\eta) \in D_{f(x)} \cap [T_x f(T_x B)]^\circ$, so it vanishes. This proves the claim.

Let $((T_x f(v),\eta),v) \in f^! D$ lie in the kernel of ψ. Then $(v,(T_x f)^*\eta) = 0$, and in particular $(T_x f(v),(T_x f)^*\eta) = 0$. But $(T_x f(v),(T_x f)^*\eta)$ is the image of $(T_x f(v),\eta) \in f^\sharp L$ under the map ϕ above. The injectivity of ϕ implies that $(T_x f(v),\eta) = 0$. Hence ψ is injective.

The following corollary follows immediately from Proposition 6.3.1 and Proposition 6.4.1.

Corollary 6.4.4 *Let D be a Dirac structure over M, transverse to f. Denote by $\mathcal{F}_D := \sharp_L(\Gamma_{\mathrm{c}}(D))$ the singular foliation of D, and $\mathcal{F}_{\mathfrak{B}_f D} := \sharp_{\mathfrak{B}_f D}(\Gamma_{\mathrm{c}}(\mathfrak{B}_f D))$ the singular foliation of the Dirac structure $\mathfrak{B}_f D$.*

Then $\mathcal{F}_{\mathfrak{B}_f D} = f^{-1}\mathcal{F}_D$.

6.5 Blowups

To match the notation of the previous sections, let

$$B := \mathrm{Blup}(M,N)$$

denote the real projective blowup along a closed and embedded submanifold $N \subseteq M$. To shorten notation, we write

$$\mathbb{P} = p^{-1}(N),$$

where $p\colon \mathrm{Blup}(M,N) \to M$ is the blowdown map. Recall that $p\colon B \setminus \mathbb{P} \to M \setminus N$ is a diffeomorphism.

6.5.1 Lifted Dirac structures on the blowup

Let D be a Dirac structure over M for which N is a transversal, i.e. $\sharp(D_y) + T_y N = T_y M$ for all $y \in N$. Via the blowdown map, one can lift D to a Dirac structure on $B \setminus \mathbb{P}$. Thanks to the transversality assumption it extends to a (unique) Dirac structure D_{Blup} on B, namely $D_{\mathrm{Blup}} = \mathfrak{B}_p D$. (More general results on lifts of Dirac structures via blowups are contained in [MSZ25]).

At every $x \in p^{-1}(N)$, the derivative of the blowdown map p satisfies $T_{p(x)}N \subseteq T_x p(T_x B)$. Hence, the map p is transverse to D, and the assumptions of Corollary 6.4.4 are satisfied. This yields the following corollary:

Corollary 6.5.1 *In the above set-up, denote by* $\mathcal{F}_D := \sharp(\Gamma_c(D))$ *the singular foliation of* D *and by* $\mathcal{F}_{\mathrm{Blup}} := \sharp_{\mathrm{Blup}}(\Gamma_c(D_{\mathrm{Blup}}))$ *the singular foliation of lifted Dirac structure* D_{Blup}*. Then*

$$\mathcal{F}_{\mathrm{Blup}} = p^{-1}\mathcal{F}_D.$$

6.5.2 Blowups and Lie algebroids

Let $\pi\colon A \to M$ be a Lie algebroid over M for which N is a transversal. Via the blowdown map, one can lift A to a Lie algebroid structure on $B \setminus \mathbb{P}$. In general, there are several distinct extensions to Lie algebroids over the whole of B. This is in contrast to the case of Dirac structures, where the uniqueness is forced by the fact that Dirac structures over B are subbundles of a prescribed vector bundle, namely $TB \oplus T^*B$. In this subsection, we want to describe the singular foliations of some of them.

One of them is the pullback Lie algebroid $p^!A$ over B, which exists, since p is transverse to A. By Proposition 6.3.1, it induces the singular foliation $p^{-1}\mathcal{F}_A$, where $\mathcal{F}_A$ is the singular foliation of A.

Other such extensions are given by the blowup of the Lie algebroid A along a Lie subalgebroid L supported on N [GL13, DS21, Obs21], see Section 1.1.4. As long as $\ker(\sharp\big|_{A|_N}) \subseteq L$, one has

$$\mathrm{Blup}(A, L) = \mathrm{Blup}(p^!A, \pi_{\mathbb{P}}^!L), \tag{6.5}$$

where the submersion $\pi_{\mathbb{P}}\colon \mathbb{P} \to N$ is the restriction of p. This is a straightforward generalisation of [Sch24, Proposition 5.16] (Proposition 3.5.16), where the case $L = \iota_N^!A$ is treated. The singular foliation of $\mathrm{Blup}(A, L)$ is necessarily tangent to the exceptional divisor $\mathbb{P}$, hence, it is distinct from $p^{-1}\mathcal{F}_A$. In particular, the Lie algebroid $\mathrm{Blup}(A, L)$ is not isomorphic to $p^!A$, but comes with a canonical Lie algebroid morphism $\mathrm{Blup}(A, L) \to p^!A \to A$.

Blowing up along the restricted Lie algebroid

A distinguished Lie subalgebroid of A over N is $\iota_N^!A = \sharp^{-1}(TN)$, which is well-defined thanks to the transversality requirement (here, $\iota_N\colon N \hookrightarrow M$ denotes the inclusion). By (6.5) we have

$$\mathrm{Blup}(A, \iota_N^!A) = \mathrm{Blup}(p^!A, \iota_{\mathbb{P}}^!p^!A),$$

whose sections are given [GL13] by

$$\{s \in \Gamma(p^!A) : \sharp_{p^!A}(s)\big|_{\mathbb{P}} \text{ is tangent to } \mathbb{P}\}. \tag{6.6}$$

To describe the singular foliation of this Lie algebroid, denote

$$\mathcal{F}_{\mathrm{tang}} := \{Y \in \mathcal{F} : Y\big|_N \text{ is tangent to } N\}.$$

Recall that every vector field tangent to N admits a (unique) lift to a vector field $\widetilde{Y}$ on B which is p-related to Y, see Lemma 4.2.1.

Proposition 6.5.2 *Let A be a Lie algebroid over M for which N is a transversal. Denote by $\mathcal{F}_{\mathrm{Blup}}$ the singular foliation of the Lie algebroid* $\mathrm{Blup}(A, \iota_N^! A)$*. Then*

$$\mathcal{F}_{\mathrm{Blup}} = \mathrm{span}_{\mathscr{C}^\infty_c(B)}\{\widetilde{Y} : Y \in \mathcal{F}_{\mathrm{tang}}\}.$$

The rest of this section is devoted to the proof of Proposition 6.5.2.

Remark 6.5.3 Let $y \in N$. By the splitting theorem for singular foliations ([AS09, Section 3.1], see also [AZ13, Proposition 2.4]) there is an open neighborhood $\mathcal{L}$ of y in the corresponding leaf of $\mathcal{F}$, a small submanifold S through y such that $T_yS \oplus T_y\mathcal{L} = T_yM$ (which we can choose such that $S \subseteq N$), and a neighborhood U of y in M such that

$$(U, \mathcal{F}_A\big|_U) \simeq (S, \iota_S^{-1}\mathcal{F}_A) \times (\mathcal{L}, \Gamma_c(T\mathcal{L})).$$

Notice that $\iota_S^{-1}\mathcal{F}$ is a singular foliation on S [AS09], hence it admits a finite local set of generators. Shrinking $\mathcal{L}$ if necessary, we can find a set of generators of $\Gamma_c(T\mathcal{L})$ given by coordinate vector fields for some chart on $\mathcal{L}$. We can assume that $N \cap U = S \times \mathcal{L}'$ for a submanifold $\mathcal{L}'$ of $\mathcal{L}$ through y, and choose the above coordinates on $\mathcal{L}$ to be adapted to $\mathcal{L}'$. Then

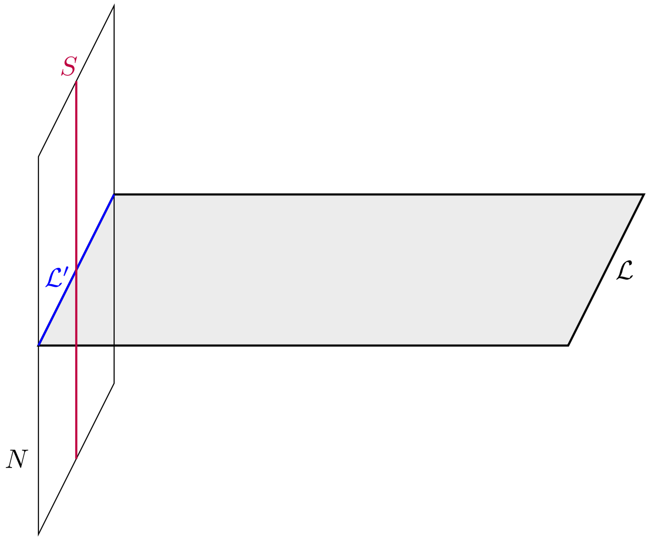

Figure 6.1: The submanifold N, the open subset $\mathcal{L}$ of a leaf, and the slice S.

i) $\mathcal{F}_A\big|_U$ admits a (finite) set of generators $\{W_i\}_{i\in I} \cup \{Z_j\}_{j\in J}$, where the W_i are tangent to N, and the Z_j are coordinate vector fields on $\mathcal{L}$ "normal" to $\mathcal{L}'$. (We can choose the $\{W_i\}_{i\in I}$ to consist of generators of $\iota_S^{-1}\mathcal{F}_A$ and coordinate vector fields on $\mathcal{L}$ which are tangent to $\mathcal{L}'$.)

ii) $\mathcal{F}_{\mathrm{tang}}\big|_U$ admits a finite set of generators, consisting of the above $\{W_i\}_{i\in I}$, and products of coordinate vector fields Z_j on $\mathcal{L}$ normal to $\mathcal{L}'$ with coordinate functions on $\mathcal{L}$ vanishing on $\mathcal{L}'$.

Item ii) shows that $\mathcal{F}_{\mathrm{tang}}\big|_U$ is a singular foliation, as it is clearly involutive.

Recall that

$$p^{-1}\mathcal{F}_{\mathrm{tang}} := \{X \in \Gamma_{\mathrm{c}}(TB) : p_*X = \sum h_i(Y_i \circ p) \text{ for } h_i \in \mathscr{C}^\infty_{\mathrm{c}}(B), Y_i \in \mathcal{F}_{\mathrm{tang}}\}. \tag{6.7}$$

A simpler description is provided by the following lemma.

Lemma 6.5.4 $p^{-1}\mathcal{F}_{\mathrm{tang}} = \mathrm{span}_{\mathscr{C}^\infty_{\mathrm{c}}(B)}\{\widetilde{Y} : Y \in \mathcal{F}_{\mathrm{tang}}\}$.

Proof. The inclusion "$\supseteq$" is satisfied since $p_*\widetilde{Y} = Y \circ p$. For the converse, take $X \in p^{-1}\mathcal{F}_{\mathrm{tang}}$ as in (6.7) (in particular, we have $Y_i \in \mathcal{F}_{\mathrm{tang}}$). Consider $\sum h_i\widetilde{Y}_i \in \Gamma_{\mathrm{c}}(TB)$. It agrees with X, since it has the same image under $p_*\colon TB \mapsto p^*TM$ and since p is a diffeomorphism on an open dense subset. Hence, X is a $\mathscr{C}^\infty_{\mathrm{c}}(B)$-linear combination of lifts of elements of $\mathcal{F}_{\mathrm{tang}}$. ☐

We now return to the Lie algebroid $\mathrm{Blup}(A, \iota_N^! A)$. The induced singular foliation $\mathcal{F}_{\mathrm{Blup}}$ is

$$\begin{aligned}\mathcal{F}_{\mathrm{Blup}} =& \sharp_{p^!A}(\Gamma_{\mathrm{c}}(p^!A)) \cap \Gamma(T^b_{\mathbb{P}}B)\\ =&(p^{-1}\mathcal{F}) \cap \Gamma(T^b_{\mathbb{P}}B)\\ =&\{X \in \Gamma_{\mathrm{c}}(TB) : p_*X = \sum h_i(Y_i \circ p) \text{ for } h_i \in \mathscr{C}^\infty_{\mathrm{c}}(B),\\ & \qquad Y_i \in \mathcal{F} \text{ s.t. } (p_*X)\big|_{\mathbb{P}} \subseteq (p\big|_{\mathbb{P}})^*TN\}. \end{aligned} \tag{6.8}$$

Here, $T^b_{\mathbb{P}}B = \mathrm{Blup}(TB, T\mathbb{P})$ denotes the log-tangent bundle of B with respect to the hypersurface $\mathbb{P}$ (its sections are the vector fields tangent to $\mathbb{P}$). The first equality holds by Equation (6.6), the second by Proposition 6.3.1, and the third because at every $x \in \mathbb{P}$, the tangent space $T_x\mathbb{P}$ is the preimage of $T_{p(x)}N$ under the derivative Tp.

Proof of Proposition 6.5.2. Thanks to Lemma 6.5.4 we have to prove that $\mathcal{F}_{\mathrm{Blup}} = p^{-1}\mathcal{F}_{\mathrm{tang}}$, comparing (6.8) and (6.7). The inclusion "$\supseteq$" is immediate.

For the converse, let $X \in \mathcal{F}_{\mathrm{Blup}}$. Since this vector field has compact support, by a partition of unity argument, it suffices to work locally on a sufficiently small open subset of B. Choose an open subset U of M as in Remark 6.5.3. Picking coordinates (x, y) on U adapted to N (i.e. $N \cap U = \{x = 0\}$) to obtain coordinates $(\tilde{x}, y)$ on $U_r \subseteq p^{-1}(U)$ as in Remark 4.2, for some index $r \in \{1, \dots, \mathrm{codim}\,(N)\}$. Recall that $\mathbb{P} \cap U_r = \{\tilde{x}_r = 0\}$. We have

$$p_*X = \sum_{i\in I} h_i(W_i \circ p) + \sum_{j\in J} k_j(Z_j \circ p),$$

where $W_i, Z_j \in \mathcal{F}_A\big|_U$ are as in Remark 6.5.3 i), $h_i \in \mathscr{C}^\infty_{\mathrm{c}}(U_r)$, and $k_j \in \mathscr{C}^\infty_{\mathrm{c}}(U_r)$ vanishes on $\mathbb{P} \cap U_r$. We have $\tilde{x}_r = p^*x_r$ for the coordinate function x_r vanishing on N. Hence, $k_j = p^*x_r \cdot k'_j$ for some $k'_j \in \mathscr{C}^\infty_{\mathrm{c}}(U_r)$. So

$$p_*X = \sum_{i\in I} h_i(W_i \circ p) + \sum_{j\in J} k'_j(x_rZ_j \circ p).$$

Since all of the $\{W_i\}_{i\in I}$ and $\{x_r Z_j\}_{j\in J}$ lie in $\mathcal{F}_{\mathrm{tang}}\big|_U$, we conclude that $X \in p^{-1}\mathcal{F}_{\mathrm{tang}}$. □

Blowing up along the isotropy Lie algebroid

Above we considered the maximal Lie subalgebroid of A supported on N, namely $\iota_N^! A$. Now we make the assumption that the isotropy Lie algebras of A have constant rank along N, and consider the resulting Lie subalgebroid $D := \ker(\sharp_A\big|_N)$. (Notice that when N is a point, the two Lie subalgebroids agree.)

Define[1]

$$\mathcal{F}_0 := \{Y \in \mathcal{F}_A : Y\big|_N = 0\}.$$

Then we have:

Proposition 6.5.5 *Let A be a Lie algebroid over M for which N is a transversal. Denote by $\mathcal{F}_{\mathrm{Blup}}$ the singular foliation of the Lie algebroid* $\mathrm{Blup}(A, L)$, *where $D := \ker(\sharp_A\big|_N)$ is assumed to have constant rank. Then*

$$\mathcal{F}_{\mathrm{Blup}} = \mathrm{span}_{\mathscr{C}^\infty_{\mathrm{c}}(B)}\{\widetilde{Y} : Y \in \mathcal{F}_0\}.$$

The proof of Proposition 6.5.5 is analog to the one of Proposition 6.5.5, hence we will just present a sketch of the arguments.

Recall that we denoted by $\pi_{\mathbb{P}}\colon \mathbb{P} \to N$ the restriction of p. By (6.5), writing out

$$\begin{aligned}\pi_{\mathbb{P}}^! L &= \{(a, X) \in \pi_{\mathbb{P}}^* L \times T\mathbb{P} : \sharp_A(a) = T\pi_{\mathbb{P}}(X)\}\\ &= \pi_{\mathbb{P}}^* L \times \ker(T\pi_{\mathbb{P}})\\ &= \sharp_{p^! A}^{-1}(\ker(T\pi_{\mathbb{P}})),\end{aligned}$$

we have

$$\mathrm{Blup}(A, L) = \mathrm{Blup}(p^! A, \sharp_{p^! A}^{-1}(\ker(T\pi_{\mathbb{P}}))).$$

Its sections are given [GL13] by

$$\{s \in \Gamma(p^! A) : \sharp_{p^! A}(s)\big|_{\mathbb{P}} \subseteq \Gamma(\ker(T\pi_{\mathbb{P}}))\}. \tag{6.9}$$

The induced foliation $\mathcal{F}_{\mathrm{Blup}}$ is

$$\mathcal{F}_{\mathrm{Blup}} = \sharp_{p^! A}(\Gamma_{\mathrm{c}}(p^! A)) \cap \Gamma(E) \tag{6.10}$$

$$= (p^{-1}\mathcal{F}) \cap \Gamma(E) \tag{6.11}$$

$$= \{X \in \Gamma_{\mathrm{c}}(TB) : p_* X = \sum h_i (Y_i \circ p) \text{ for } h_i \in \mathscr{C}^\infty_{\mathrm{c}}(B),\ Y_i \in \mathcal{F}_A \text{ s.t. } (p_* X)\big|_{\mathbb{P}} = 0\}. \tag{6.12}$$

Here, E is the **edge Lie algebroid** [Fin22] associated to the fibration $\pi_{\mathbb{P}}$ of the hypersurface $\mathbb{P}$; its sections are vector fields on B which over $\mathbb{P}$ are tangent

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[1] Even though it will not be used in our arguments, we point out that $\mathcal{F}_0$ is locally finitely generated. Local generators are obtained multiplying the vector fields $\{W_i\}_{i\in I}$ and $\{Z_j\}_{j\in J}$ in Remark 6.5.3 with coordinate functions on $\mathcal{L}$ vanishing on $\mathcal{L}'$.

to the fibres, i.e. lie in $\ker(T\pi_{\mathbb{P}})$. The first equality holds by Equation (6.9), the second by Proposition 6.3.1 and the third because of the definition of the edge Lie algebroid.

Recall that

$$p^{-1}\mathcal{F}_0 := \{X \in \Gamma_{\mathrm{c}}(TB) : p_*X = \sum h_i(Y_i \circ p) \text{ for } h_i \in \mathscr{C}^\infty_{\mathrm{c}}(B), Y_i \in \mathcal{F}_0\}. \tag{6.13}$$

The analogue of Lemma (6.5.4) holds, i.e. $p^{-1}\mathcal{F}_0 = \mathrm{span}_{\mathscr{C}^\infty_{\mathrm{c}}(B)}\{\widetilde{Y} : Y \in \mathcal{F}_0\}$.

Given this, to prove Proposition 6.5.5 we have to show that $\mathcal{F}_{\mathrm{Blup}} = p^{-1}\mathcal{F}_0$ using (6.12) and (6.13). This is done as in Proposition 6.5.2, the only difference being that the coefficients h_i and k_j appearing there all vanish on $\mathbb{P}$.

Remark 6.5.6 Taking $A = TM$ the tangent Lie algebroid, from Equation (6.12) and Proposition 6.5.5 we obtain

$$\Gamma_{\mathrm{c}}(E) = \mathrm{span}_{\mathscr{C}^\infty_{\mathrm{c}}(B)}\{\widetilde{Y} : Y \in \Gamma_{\mathrm{c}}(TM) \text{ vanishes on } N\}.$$

In other words, all (compactly supported) sections of the edge Lie algebroid associated to the fibration $\pi_{\mathbb{P}}$ are obtained as lifts of vector fields vanishing on N.

Remark 6.5.7 More generally, let $D \subseteq A$ be a Lie subalgebroid supported over N that contains the isotropy bundle $\ker(\sharp_A\big|_N)$. Denote

$$\mathcal{F}_{A,L} := \{Y \in \mathcal{F}_A : Y\big|_N \in \sharp_A(\Gamma_{\mathrm{c}}(L))\}.$$

Then analog to Proposition 6.5.2 ($\mathcal{F}_{A,\iota_N^! A} = \mathcal{F}_{\mathrm{tang}}$) and Proposition 6.5.5 ($\mathcal{F}_{A,L} = \mathcal{F}_0$ one can show that

$$\mathcal{F}_{\mathrm{Blup}} = \mathrm{span}_{\mathscr{C}^\infty_{\mathrm{c}}(B)}\{\tilde{Y} : Y \in \mathcal{F}_{A,L}\}. \tag{6.14}$$

Indeed, in (6.10) and (6.11) replace $\Gamma(E)$ by

$$\mathcal{E}_L := \{X \in \Gamma_{\mathrm{c}}(TB) : X\big|_{\mathbb{P}} \in \pi_{\mathbb{P}}^{-1}(\sharp_A(\Gamma_{\mathrm{c}}(L)))\},$$

and at the end of (6.12) write $(p_*X)\big|_{\mathbb{P}} \in \pi_{\mathbb{P}}^{-1}(\sharp_A(\Gamma_{\mathrm{c}}(L)))$.

Since the analog statement to Lemma 6.5.4 holds, to show (6.14) it is enough to prove

$$\mathcal{F}_{\mathrm{Blup}} = p^{-1}\mathcal{F}_{A,L}.$$

The inclusion "$\supseteq$" is clear. For the converse, pick

- a tubular neighbourhood $\mathrm{pr}\colon V \to N$ of N in M with $A\big|_V \simeq \mathrm{pr}^!\iota_N^! A$ (which exists by [BLM16, Theorem 4.1]),
- a vector bundle complement C satisfying $\iota_N^! A = L \oplus C$,
- a horizontal lift $\cdot^{\mathrm{hor}}\colon \Gamma(TN) \to \Gamma(TV)$.

Then sections of $A\big|_V$ are generated by sections of the form

$$\begin{aligned} &(0, X), && \text{for } X \text{ a vertical vector field on } V, \\ &(a \circ \mathrm{pr}, \sharp_A(a)^{\mathrm{hor}}), && \text{for } a \in \Gamma(L) \text{ or } a \in \Gamma(C). \end{aligned}$$

The argument of the proof of Proposition 6.5.2 goes through, where sections of C under the anchor and vertical vector fields replace the Z_j, and sections of D under the anchor replace the W_i.

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Chapter 7

Conclusion

We first introduced two methods to compute Lie algebroid cohomology. The Serre spectral sequence from Chapter 2 is a more direct computational tool. It gives a unifying framework for many spectral sequences known in the literature, such as Mackenzie's spectral sequence for Lie algebroid extensions, the Hochschild-Serre spectral sequence for Lie algebras, or the Leray-Serre spectral sequence for locally trivial fibre bundles.

Moreover, we used the Serre spectral sequence to generalise results from horizontally nondegenerate Poisson structures to Dirac structures, and in Chapter 3 we prove a Gysin long exact sequence for Lie algebroids.

So far, all interesting examples have come from the Serre spectral sequence induced by the inclusion of a Lie subalgebroid. However, as remarked in Chapter 3, it is possible to consider more general Lie algebroid morphisms.

The second tool we developed for Lie algebroid cohomology makes use of real projective blowups. In Chapter 3 we studied the blowdown map in cohomology. On one hand, we make use of the understanding of the blowdown map in cohomology by expressing the cohomology of $\operatorname{Blup}(A, \iota^! A)$ in terms of $\mathrm{H}^\bullet(A)$, where $\iota\colon N \hookrightarrow M$ is a transversal of a Lie algebroid $A \Rightarrow M$. On the other hand, we use the real projective blowup to desingularise the action Lie algebroid $\mathfrak{so}(3) \ltimes \mathbb{R}^3$, compute the cohomology of the blowup via spectral sequences, and deduce the cohomology of $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ via our studies of the blowdown map.

There exist, aside from real projective blowups, more constructions for desingularising Lie algebroids, like spherical blowups [DS21], or Nash-type blowups [Lou24]. One can hope that the combination of desingularisation and understanding the blowdown map in cohomology in these settings, too, might lead to new ways of computing Lie algebroid cohomologies.

Inspired by Polishchuk's result on the lift of Poisson structures [Pol97], for which we give an alternative proof in Chapter 5, we then turned to discussing the lift of Dirac structures to real projective blowups in Chapter 4. We found an if-and-only-if condition on the Dirac structure and the submanifold for the lift to exist. These conditions are, considering the generality of the problem, quite restrictive. As discussed in Section 4.1, it might be possible to obtain more flexibility using different kinds of blowup procedures.

Of course, everything related to blowups can also be considered in the context of complex geometry and the blowup procedures therein.

In Chapter 6 we discussed the relations between pullbacks of Lie algebroids, Dirac structures, and their underlying singular foliations. We gave proof for the fact that taking pullbacks commutes with both viewing a Dirac structure as a Lie algebroid as well as taking the singular foliation underlying a Dirac or Lie algebroid structure. It is known in the field that these operations commute, however there seem to exist few written accounts of it in the literature.

Finally, we discussed the blowup by a transversal $\iota\colon N \hookrightarrow M$ of a Dirac or Lie algebroid structure with the focus on their singular foliation. For a Dirac structure $D \subseteq TM \oplus T^*M$, the requirement of being a subbundle of the generalised tangent bundle forces a lift to the blowup to be unique (namely, $\mathfrak{B}_p D$), see also Chapter 4. For a Lie algebroid $A \Rightarrow M$, on the other hand, we can consider the pullback $p^! A$ of Lie algebroids via the blowdown map, but also the Lie algebroid blowup $\mathrm{Blup}(A, L)$ for any Lie subalgebroid $L \Rightarrow N$. Both constructions give meaningful lifts of A to $\mathrm{Blup}(M, N)$, which are not isomorphic. For L containing the isotropies over N, i.e. $\ker(\sharp|_{A_N}) \subseteq L$, we characterised the singular foliation of $\mathrm{Blup}(A, L)$ both in terms of the singular foliation of $p^! A$ and A itself.

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About the cover

The image decorating this thesis is artwork created by my sister, Daniela Schüßler. Daniela, your paintings have been all around the house when I grew up, your art has decorated my apartments for years, and now I am more than happy to see it cover my thesis, too.

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About the author

Andreas Schüßler was born in Aschaffenburg, Germany, on October 16, 1996. He obtained his highschool diploma (Abitur) at HSG Hösbach in 2015. After that, he began studying mathematical physics in Würzburg.

In 2018 Andreas obtained his Bachelor degree and in 2021 his Master degree, both in mathematical physics under the supervision of prof. dr. Stefan Waldmann.

Between September 2021 and August 2025 he was a PhD student at Radboud Universiteit Nijmegen and KU Leuven under the supervision of prof. dr. Erik Koelink, prof. dr. Ioan Mărcuț, and prof. dr. Marco Zambon. He moved from Nijmegen to Leuven in August 2023.